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**TIP-POSITION CONTROL OF A FLEXIBLE BEAM: MODELING
APPROACHES AND EXPERIMENTAL VERIFICATION**

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INTRODUCTION

The dynamic modeling part of an experimental/theoretical effort to develop and test pointing and tracking control techniques for a mechanical system containing both rigid and flexible bodies is presented in this report. Recently, much work was done to model open-loop chains of rigid and elastic bodies with application to kinematic linkages, spacecraft, and manipulators.

In this modelling effort, the work of Schmitz¹ was used as a guide. The mechanical system consisted of a flexible beam fixed to a rigid disk and rotating in the horizontal plane. The new contributions that result from this work are: (1) the inclusion of the size (the radius of the rigid disk) as well as the mass properties of the rigid moving base in deriving the equations of motion, (2) the study of the effect of the size of the moving rigid base on the dynamic response of the system, and (3) the development and comparison of several modelling approaches which can be used for control design: a reduced-order model with cantilever modes, a reduced-order model with system modes, a model based on finite element analysis, and a lumped parameter model.

The computer software used for this work was: MSC Pa12 (finite element analysis), MATRIXx (dynamic analysis and control design), and MathCAD (multipurpose engineering software). The book of Meirovitch² was referred to often during the course of this work.

Work currently engaged in is: (1) completion of model validation, including study of the effects of Coulomb and viscous damping; (2) development and testing of several control algorithms: PID, LQG/LTR, adaptive and H_∞ ; and (3) addition of an end-point position sensor to the experimental apparatus to allow for end-point position feedback control.

¹Schmitz, E., "Experiments on the End-Point Position Control of a Very Flexible One-Link Manipulator," Ph.D. Thesis, Stanford University, Guidance and Control Laboratory, Department of Aeronautics and Astronautics, Stanford, CA, 1985.

²Meirovitch, L., Analytical Methods in Vibrations, The Macmillan Company, New York, 1967.

EXPERIMENTAL APPARATUS

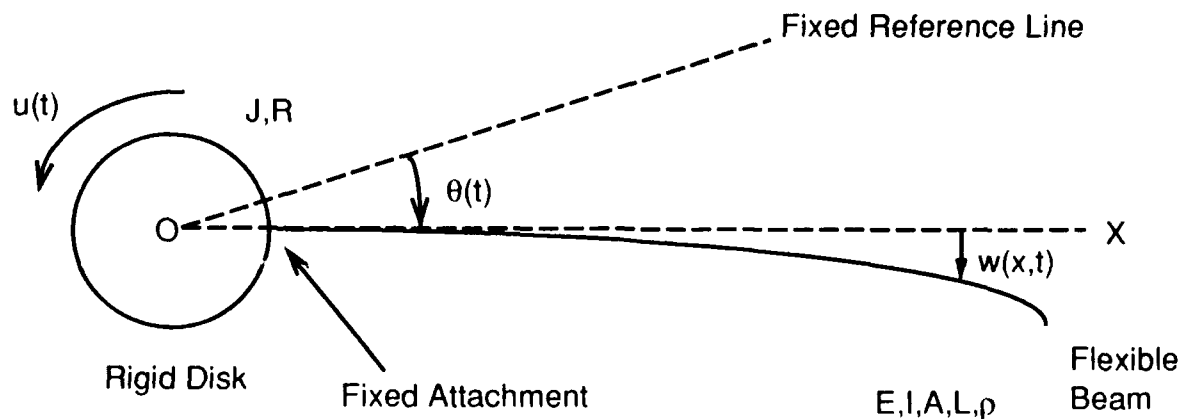
An experimental apparatus (fig 1) was built to demonstrate pointing and tracking control strategies for fast-moving, lightweight, flexible appendages to rigid bodies. It consists of a one degree of freedom rigid disk, rotating about a vertical axis, with a long, thin, flexible beam fixed to it in the horizontal plane. Two dc torque motors, one used for disturbance input and one used for control, cause the disk to rotate about the vertical axis. A resolver is used to measure disk angle, which is the only sensor on the system at the present time. A beam end-point position sensor is presently being added to the system.

DYNAMIC MODELLING

Dynamic equations were developed for a mechanical system consisting of a flexible beam fixed to a rigid disk. Goals were to understand the fundamental characteristics of the system and to develop a model of the system which can be used to test new control algorithms.

Model Description and Derivation of Equations of Motion

Consider a uniform flexible beam of length L , fixed to a rigid disk of radius R , and moving in the horizontal plane, as shown in the figure below.



- $u(t)$ = externally applied torque
- J = mass moment of inertia of rigid disk about axis of rotation
- R = radius of rigid disk
- E = modulus of elasticity of beam material
- I = area moment of inertia of beam cross section about neutral axis
- A = cross-sectional area of beam
- ρ = density of beam material $\sigma = \rho \cdot A$ = mass per unit length
- L = length of beam

Displacement of any point P along the beam's neutral axis at a distance x from the disk is given by the arbitrarily large disk angle $\theta(t)$ and the small elastic deflection $w(x,t)$ measured from the line OX. The assumptions made in deriving the equations of motion are:

- Neglect axial deformations and axial forces. The dynamic stiffening effect and increased natural frequencies that result from the axial forces are assumed negligible in this situation.

- Contribution of the rotational moment of inertia of an elementary beam section to the kinetic energy of the system is neglected. This is valid if the cross-sectional dimensions of the beam are small compared to its length.

- Effects of shear deformation are negligible. This is valid if the cross-sectional dimensions of the beam are small compared to its length and if the frequency of beam oscillations is low.

- Elastic deflection of the beam and angular velocity of the disk are small.

- Neglect any energy dissipation. This will be added later in the analysis.

Expressions for the kinetic energy, T , and the potential energy, V , of the mechanical system are as follows:

$$T = \frac{1}{2} \cdot J \cdot \left[\frac{d}{dt} \theta(t) \right]^2 + \frac{1}{2} \cdot \int_0^L \sigma \left[(R+x) \cdot \frac{d}{dt} \theta(t) + \frac{d}{dt} w(x,t) \right]^2 dx$$

$$V = \frac{1}{2} \cdot \int_0^L E \cdot I \cdot \left[\frac{d^2}{dx^2} w(x,t) \right]^2 dx - U(t) \cdot \theta(t)$$

Given the expressions for the kinetic and potential energy, the Hamilton's principle was applied:

$$\delta \cdot \int_{t_1}^{t_2} (T - V) dt = 0$$

The equation of motion and boundary conditions that result are:

Equation of motion:

$$E \cdot I \cdot \left[\frac{d^4}{dx^4} w(x,t) \right] + \sigma \cdot \left[(R+x) \cdot \left[\frac{d^2}{dt^2} \theta(t) \right] + \left[\frac{d^2}{dt^2} w(x,t) \right] \right] = 0$$

Boundary conditions

- $J \cdot \left[\frac{d^2}{dt^2} \cdot \theta(t) \right] - U(t) - E \cdot I \cdot \left[\frac{d^2}{dx^2} \cdot w(x,t) \right]_{x=0} + R \cdot E \cdot I \cdot \left[\frac{d^3}{dx^3} \cdot w(x,t) \right]_{x=0} = 0$
- $w(x,t) = 0$ at $x = 0$ (deflection at fixed connection = 0)
- $\frac{d}{dx} \cdot w(x,t) = 0$ at $x = 0$ (slope at fixed connection = 0)
- $\frac{d^2}{dx^2} \cdot w(x,t) = 0$ at $x = L$ (moment at free end = 0)
- $\frac{d^3}{dx^3} \cdot w(x,t) = 0$ at $x = L$ (shear force at free end = 0)

Definition of a new coordinate: $y(x,t) = (R + x) \cdot \theta(t) + w(x,t)$ = inertial lateral displacement of the beam at a distance x from the edge of the disk.

The equation of motion and boundary conditions reduce to the following:

$$E \cdot I \cdot \left[\frac{d^4}{dx^4} \cdot y(x,t) \right] + \sigma \cdot \left[\frac{d^2}{dt^2} \cdot y(x,t) \right] = 0$$

- $J \cdot \left[\frac{d^2}{dt^2} \cdot \theta(t) \right] - U(t) - E \cdot I \cdot \left[\frac{d^2}{dx^2} \cdot y(x,t) \right]_{x=0} + R \cdot E \cdot I \cdot \left[\frac{d^3}{dx^3} \cdot y(x,t) \right]_{x=0} = 0$
- $y(x,t) = R \cdot \theta(t)$ at $x = 0$
- $\frac{d}{dx} \cdot y(x,t) = \theta(t)$
- $\frac{d^2}{dx^2} \cdot y(x,t) = 0$ at $x = L$
- $\frac{d^3}{dx^3} \cdot y(x,t) = 0$ at $x = L$

The following section show several ways to solve these equations in a form which is useful for feedback control systems.

Exact Solution and Open-Loop Transfer Functions

The variables $y(x,s)$, $u(s)$, and $\theta(s)$ are defined as the Laplace transforms of the variables $y(x,t)$, $u(t)$, and $\theta(t)$, respectively. Taking the Laplace transform of the equation of motion and boundary conditions, the following is obtained:

Equation of Motion

$$\left[\frac{d^4}{dx^4} \cdot y(x,s) \right] + \frac{\sigma \cdot s^2}{E \cdot I} \cdot y(x,s) = 0$$

Boundary Conditions:

- $J \cdot s^2 \cdot \theta(s) - u(s) - E \cdot I \cdot \left[\frac{d^2}{dx^2} \cdot y(x,s) \right]_{x=0} + R \cdot E \cdot I \cdot \left[\frac{d^3}{dx^3} \cdot y(x,s) \right]_{x=0} = 0$
- $y(x,s) = R \cdot \theta(s)$ at $x = 0$
- $\frac{d}{dx} \cdot y(x,s) = \theta(s)$ at $x = 0$
- $\frac{d^2}{dx^2} \cdot y(x,s) = 0$ at $x = L$
- $\frac{d^3}{dx^3} \cdot y(x,s) = 0$ at $x = L$

The following dimensionless ratios are defined:

$$\frac{R}{L} = \delta \quad \frac{J}{\sigma \cdot L^3} = \theta$$

Note that $\sigma \cdot L^3 = (\sigma \cdot L) \cdot L^2 = m \cdot L^2$ where m is the total mass of the beam. The dimensionless complex number λ related to the Laplace variable s is defined by:

$$\beta^4 = \frac{[\sigma \cdot s^2]}{E \cdot I} \quad \lambda = \beta \cdot L$$

The general solution of the equation of motion is:

$$y(x,s) = A \cdot \sin(\beta \cdot x) + B \cdot \sinh(\beta \cdot x) + C \cdot \cos(\beta \cdot x)$$

With this equation and the boundary conditions stated above, the following is found:

$$M(\lambda) \cdot \begin{bmatrix} A \\ C \\ \theta \end{bmatrix} = \frac{1}{E \cdot I \cdot \beta} \cdot \begin{bmatrix} 0 \\ 0 \\ -u(s) \end{bmatrix} \quad \begin{aligned} D &= R \cdot \theta - C \\ B &= \frac{\theta}{\beta} - A \end{aligned}$$

$$\begin{bmatrix} [-\sin(\lambda) - \sinh(\lambda)] & [-\cos(\lambda) - \cosh(\lambda)] & R \cdot \cosh(\lambda) + \frac{\sinh(\lambda)}{\beta} \\ [-\cos(\lambda) - \cosh(\lambda)] & [\sin(\lambda) - \sinh(\lambda)] & \frac{\cosh(\lambda)}{\beta} + R \cdot \sinh(\lambda) \\ 2 \cdot \delta \cdot \lambda & -2 & \frac{\varepsilon \cdot \lambda}{\beta} - \frac{\delta \cdot \lambda}{\beta} + R \end{bmatrix} = M(\lambda)$$

Solving for A, C, and θ :

$$A = \frac{1}{K} \cdot \begin{bmatrix} \frac{-u(s)}{2} \\ E \cdot I \cdot \beta \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{\beta} \cdot [1 + \sin(\lambda) \cdot \sinh(\lambda) + \cos(\lambda) \cdot \cosh(\lambda)] \\ + (-R) [\cos(\lambda) \cdot \sinh(\lambda) + \sin(\lambda) \cdot \cosh(\lambda)] \end{bmatrix}$$

$$C = \frac{1}{K} \cdot \begin{bmatrix} \frac{u(s)}{2} \\ E \cdot I \cdot \beta \end{bmatrix} \cdot \begin{bmatrix} R \cdot [1 + \cos(\lambda) \cdot \cosh(\lambda) - \sin(\lambda) \cdot \sinh(\lambda)] \\ + \frac{1}{\beta} \cdot [\cos(\lambda) \cdot \sinh(\lambda) - \sin(\lambda) \cdot \cosh(\lambda)] \end{bmatrix}$$

$$\theta = \frac{1}{K} \cdot \frac{2 \cdot u(s)}{E \cdot I \cdot \beta} \cdot [1 + \cos(\lambda) \cdot \cosh(\lambda)]$$

$$K = \frac{-2}{\beta} \cdot \begin{bmatrix} [1 + \cos(\lambda) \cdot \cosh(\lambda)] \cdot \varepsilon \cdot \lambda^3 + \sin(\lambda) \cdot \cosh(\lambda) \cdot [1 + \lambda^2 \cdot \delta^2] \\ + 2 \cdot \lambda \cdot \delta \cdot \sin(\lambda) \cdot \sinh(\lambda) + \cos(\lambda) \cdot \sinh(\lambda) \cdot [\lambda^2 \cdot \delta^2 - 1] \end{bmatrix}$$

The following open-loop transfer functions are of interest:

$$\frac{\theta(s)}{u(s)} \quad \text{Disk angle sensor colocated with the actuator.}$$

$\frac{y(L,s)}{\theta(s)}$ Tip position sensor (now being added to experimental apparatus) non-colocated, i.e., separated from the actuator by the elastic beam.

$$\frac{\theta(s)}{u(s)} = \frac{N\theta(\lambda)}{D(\lambda)} \text{ and } \frac{y(L,s)}{u(s)} = \frac{N_{tip}(\lambda)}{D(\lambda)}$$

$$D(\lambda) = \frac{-2 \cdot L}{\lambda} \cdot \left[\begin{aligned} &[1 + \cos(\lambda) \cdot \cosh(\lambda)] \cdot \varepsilon \cdot \lambda^3 + \sin(\lambda) \cdot \cosh(\lambda) \cdot [1 + \lambda^2 \cdot \delta^2] \dots \\ &+ 2 \cdot \lambda \cdot \delta \cdot \sin(\lambda) \cdot \sinh(\lambda) + \cos(\lambda) \cdot \sinh(\lambda) \cdot \left[\frac{2}{\lambda} \cdot \delta^2 - 1 \right] \end{aligned} \right]$$

$$N\theta(\lambda) = \left[\frac{2 \cdot L}{E \cdot I \cdot \lambda} \right] \cdot [1 + \cos(\lambda) \cdot \cosh(\lambda)]$$

$$N_{tip}(\lambda) = \left[\frac{2 \cdot L}{E \cdot I \cdot \lambda} \right] \cdot (L \cdot [\sin(\lambda) + \sinh(\lambda)] + R \cdot \lambda \cdot [\cos(\lambda) + \cosh(\lambda)])$$

The roots of the function $D(\lambda)$ are the poles of the transfer functions, also called the natural vibration frequencies of the rigid disk/flexible beam system. As stated in the initial assumptions, this equation is valid only to predict the first few vibration frequencies of the system.

$D(\lambda)$, $N\theta(\lambda)$, and $N_{tip}(\lambda)$ are expressed in Taylor series expansions. For example, in the case of $D(\lambda)$, write:

$$D(\lambda) = \sum \frac{\lambda^n}{n!} \cdot \left[\frac{d^n}{d\lambda^n} \cdot D(\lambda) \right]_{\lambda=0}$$

The effect of the radius and mass moment of inertia of the rigid disk on the fundamental frequency of the rigid disk/flexible beam system are shown in figures 2 and 3. The computation of the exact open-loop transfer functions and the reduced order open-loop transfer functions using a Taylor series expansion is shown in appendix A which gives a Bode plot comparison between them. The beam tip response and disk angle response to a unit step torque applied to the disk is shown in figure 4. Reduced-order models of the open-loop transfer functions were used for this simulation with the first four vibration modes kept.

The roots of $N\theta(\lambda)$ are the same as the roots of $D(\lambda)$ for $\epsilon = 0$, i.e., for an infinitely large disk moment of inertia. Therefore, the colocated zeros [the roots of $N\theta(\lambda)$] correspond to the natural vibration frequencies of a cantilevered beam. These frequencies are also called antiresonance frequencies. If the actuator is driven with a sinusoidal command at one of these frequencies, the rigid disk will not move and the elastic beam vibrates according to the selected cantilevered mode shape.

The tip position transfer function has zeros with a positive real part; therefore, the system is a nonminimum phase system. The initial time response of the beam tip to a unit step torque input to the disk is a short quasi-stationary period (approximately 0.0066 seconds) followed by motion of the tip in the opposite direction to that of the rigid disk (fig. 4). The contributions to the tip position of the rigid body mode and the first flexible modes. This type of initial response presents a difficult control problem.

Reduced-Order Model with Cantilevered Modes

It is common to use the constrained mode shapes (cantilevered modes) rather than the unconstrained mode shapes of the system in order to attempt to describe the dynamic behavior of an elastic structure. Although the system modes lead to a very simple formulation for the dynamic equations, they are difficult to use for a multilink flexible manipulator because of their dependence on the relative orientation of the links. Formulation in terms of the cantilevered modes does not have this drawback.

The elastic deflection $w(x,t)$ is expressed in terms of the following series:

$$w(x,t) = \sum_i \phi_i(x) \cdot q_i(t)$$

The mode shapes for a cantilever beam have been previously derived in various reference books and textbooks and can be expressed as follows:

$$\begin{aligned} \phi(x) = & A \cdot [\cos(\beta \cdot x) + \cosh(\beta \cdot x)] + B \cdot [\cos(\beta \cdot x) - \cosh(\beta \cdot x)] \cdot \dots \\ & + C \cdot [\sin(\beta \cdot x) + \sinh(\beta \cdot x)] + D \cdot [\sin(\beta \cdot x) - \sinh(\beta \cdot x)] \end{aligned}$$

The boundary conditions are:

$$\text{at } x = 0: \phi(x) = 0$$

$$\text{at } x = L: \quad \frac{d}{dx} \phi(x) = 0 \quad \frac{d^2}{dx^2} \phi(x) = 0 \quad \frac{d^3}{dx^3} \phi(x) = 0$$

Therefore,

$$A = C - O \text{ and } \frac{D}{B} = - \left[\frac{\cos(\beta \cdot L) + \cosh(\beta \cdot L)}{\sin(\beta \cdot L) + \sinh(\beta \cdot L)} \right] = \left[\frac{\sin(\beta \cdot L) - \sinh(\beta \cdot L)}{\cos(\beta \cdot L) + \cosh(\beta \cdot L)} \right]$$

Therefore, the mode shapes can be expressed as:

$$\phi(x) = -L \cdot \left[\cos(\beta \cdot x) - \cosh(\beta \cdot x) - \left[\frac{\cosh(\lambda) + \cos(\lambda)}{\sinh(\lambda) + \sin(\lambda)} \right] \cdot [\sin(\beta \cdot x) - \sinh(\beta \cdot x)] \right]$$

where $\lambda = \beta L$ is a root of the equation $[1 + \cos(\lambda)\cosh(\lambda)] = 0$. The first four roots of this equation are 1.8751, 4.6941, 7.8548, and 10.9955, and the values of D/B corresponding to these values of λ are -0.7341, -1.0185, -0.9992, and -1.0000, respectively. The cantilevered frequencies are related to the values of λ by the following equation:

$$\omega^2 = \frac{E \cdot I}{\sigma \cdot L^4} \cdot \lambda^4$$

In the expressions for kinetic energy, t , and potential energy, V , in $w(x,t)$ is replaced with its series representation. The result is:

$$T = \frac{1}{2} \cdot \left[(j + F1) \cdot \dot{\theta}^2 + \dot{\theta} \cdot \sum_1 \dot{q}_i \cdot |F2_i + F3_i| + \sum_1 |\dot{q}_i|^2 \cdot F4_i \right]$$

$$V = \frac{1}{2} \cdot \sum_1 |q_i|^2 \cdot F5_i - u(t) \cdot \theta$$

$$F1 = \sigma \cdot L \cdot \left[R^2 + R \cdot L + \frac{L^2}{3} \right]$$

$$F2_i = 2 \cdot R \cdot \int_0^L \sigma \cdot \phi(x)_i \, dx$$

$$F3_i = 2 \cdot \int_0^L \sigma \cdot x \cdot \phi(x)_i \, dx$$

$$F4_i = \int_0^L \sigma \cdot |\phi(x)_i|^2 \, dx$$

$$F5_i = \int_0^L E \cdot I \cdot \left[\frac{d^2}{dx^2} \cdot \phi(x)_i \right]^2 \, dx$$

Lagrange's equations was applied:

$$\frac{d}{dt} \cdot \left[\frac{d}{dq_i} \cdot T \right] - \frac{d}{dq_i} \cdot T + \frac{d}{dq_i} \cdot V = 0$$

An infinite set of coupled ordinary differential equations was obtained. Using the first three mode shapes of the cantilever beam, the equations of motion that result are as follows:

$$(j + F1) \cdot \ddot{\theta} + \frac{1}{2} \cdot \sum_i |F2_i + F3_i| \ddot{q}_i = u(t)$$

$$\frac{1}{2} \cdot |F2_i + F3_i| \cdot \ddot{\theta} + F4_i \cdot \ddot{q}_i + F5_i \cdot q_i = 0 \quad i = 1, 2, 3$$

There are four equations and four unknowns: θ q_1 q_2 q_3 . These equations can now be put in state-space form where the state variables are:

$$\begin{array}{llll} x_1 = \theta & x_3 = q_1 & x_5 = q_2 & x_7 = q_3 \\ x_2 = \dot{\theta} & x_4 = \dot{q}_1 & x_6 = \dot{q}_2 & x_8 = \dot{q}_3 \end{array}$$

The state-variable equations that result are as follows:

$$\dot{x}_{2 \cdot i - 1} = x_{2 \cdot i} \quad i = 1, 2, 3, 4$$

$$\dot{x}_2 = \frac{\frac{1}{J + F1}}{1 - \left[\frac{1}{J + F1} \right] \cdot \sum_i \frac{|F2_i + F3_i|^2}{4 \cdot F4_i}} \cdot \left[u(t) + \sum_i \frac{F5_i}{F4_i} \cdot \frac{F2_i + F3_i}{2} \cdot x_{2 \cdot i + 1} \right]$$

$$\dot{x}_{2 \cdot i + 1} = \left[\frac{F2_i + F3_i}{2 \cdot F4_i} \right] \cdot \left[\frac{\frac{-1}{J + F1}}{1 - \frac{1}{J + F1} \cdot \sum_i \left[\frac{|F2_i + F3_i|^2}{4 \cdot F4_i} \right]} \right] \cdot$$

$$\left[u(t) + \sum \frac{F5_i}{2} \cdot \frac{F2_i + F3_i}{F4_i} \cdot x_{2 \cdot i + 1} \right] - \left[\frac{F5_i}{F4_i} \right] \cdot x_{2 \cdot i + 1} \quad i = 1, 2, 3$$

The computation of the state-variable equations and the equations of motion in state-space matrix form are shown in appendix B. Bode plots of magnitude (dB) versus frequency (rad/sec) for disk angle motion and beam tip motion are shown in figure 5. Also shown in this figure are the beam tip response and disk angle response to a unit step torque applied to the disk.

Finite Element Analysis

A finite element analysis program was used to generate a dynamic model in the following form:

$$[M] [\ddot{q}] + [K] [q] = [Q]$$

where $[M]$ is the mass matrix, $[K]$ is the stiffness matrix, $[q]$ is the nodal displacement matrix, and $[Q]$ is the external force matrix. This model is valid for small linear elastic deformation and small angular velocities of the rigid base. Using the same parameter values as given in appendices A and B, the output from the finite element analysis program is given in appendix C. The above equation can be transformed into a set of decoupled modal equations as follows. First, solve the eigenvalue problem

$$[M] [U] [\ddot{w}^2] = [K] [u]$$

where

$[u]$ = modal matrix

$[\ddot{w}^2]$ = diagonal matrix of eigenvalues

The response may be described as a superposition of normal modes in the form:

$$[q] = [u] [\beta]$$

where $[\beta(t)]$ is a column matrix consisting of a set of time-dependent generalized coordinates. Therefore,

$$[\ddot{q}] = [u] [\ddot{\beta}]$$

$$[M] [u] [\ddot{\beta}] + [K] [u] [\beta] = [Q]$$

$$[u]^T [M] [u] [\ddot{\beta}] + [u]^T [K] [u] [\beta] = [u]^T [Q]$$

But normal modes are such that

$$[u]^T [M] [u] = [I]$$

$$[u]^T [K] [u] = [\ddot{\omega}^2]$$

In addition, a column matrix of generalized forces $N(t)$ was introduced associated with the generalized coordinates $\beta(t)$ and related to the forces $Q(t)$ by

$$[N] = [u]^T [Q]$$

therefore,

$$[\dot{\beta}] + [\ddot{\omega}^2] [\beta] = [N]$$

This represents a set of n uncoupled differential equations of the type

$$\ddot{\beta}(t)_r + \ddot{\omega}^2 \cdot \beta(t)_r = N(t)_r \quad r = 1, 2, \dots, n$$

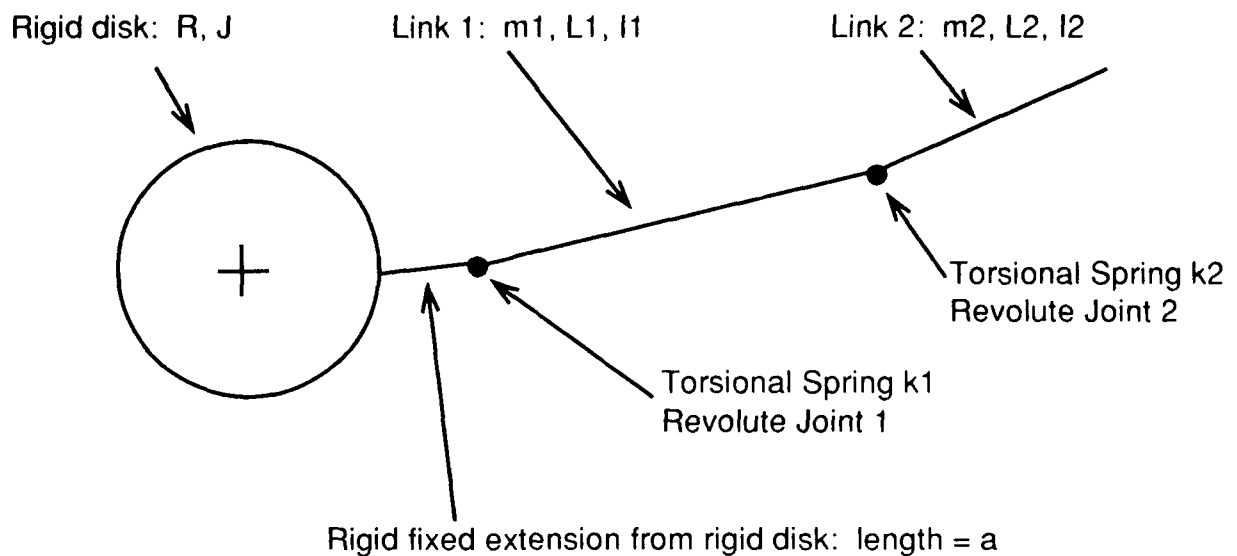
Therefore, modal analysis consists of uncoupling the equations of motion by means of a linear coordinate transformation; the transformation matrix is just the modal matrix $[u]$. Details of this procedure are given in appendix C. Using the results of this analysis, bode plots of magnitude (dB) versus frequency (rad/sec) for disk angle motion and beam tip response and the beam tip response and disk angle response to a unit step torque are shown in figure 6.

Lumped Parameter Model

P. Sheth and K. Craig have developed a rigid disk, 2 rigid link, 2 torsional spring model of the rigid disk/flexible beam dynamic system that preserves the following characteristics of the continuous system:

1. Natural frequencies of the first two modes of vibration
2. Mode shapes of the first two modes of vibration

This lumped model is a first approximation of the real structure, although it may be an excellent approximation for specific situations. A diagram of the lumped model along with its parameters is shown below.



Parameter values (units: in-lbf-sec):

$m_1 = 0.000674$	$I_1 = 0.008346$	$k_1 = 506.668$	$L_1 = 18.96$
$m_2 = 0.000478$	$I_2 = 0.003814$	$k_2 = 264.105$	$L_2 = 13.44$
$a = 3.6$			

The details of this work and its extension to other multilink dynamic systems are now in preparation for publication.

Reduced-Order Model with System Modes

A standard way to solve the dynamic equations previously given is to expand the solution $y(x,t)$ as the infinite series:

$$y(x,t) = \sum_i \phi(x)_i \cdot q(t)_i$$

where $\phi(x)$ are the eigenfunctions of the equation of motion with the boundary conditions, and $q(t)$ are the corresponding time-dependent generalized coordinates. $\phi(x)$ are called the system or unconstrained mode shapes. To find the mode shapes $\phi(x)$, the external torque $u(t)$ is set equal to zero. Then search for the eigensolutions of the equation of motion and boundary conditions of the form:

$$y(x,t)_i = \phi(x)_i \cdot e^{j \cdot \ddot{\omega}_i \cdot t}$$

We obtained a fourth order ordinary differential equation in the variable x whose general solution is:

$$\phi(x) = A \cdot \sin(\beta \cdot x) + B \cdot \sinh(\beta \cdot x) + C \cdot \cos(\beta \cdot x) + D \cdot \cosh(\beta \cdot x)$$

This equation has already been studied. The solution to the following system of equation is given in appendix D.

$$\begin{bmatrix} -\sin(\lambda) - \sinh(\lambda) & -\cos(\lambda) - \cosh(\lambda) & R \cdot \cosh(\lambda) + L \cdot \frac{\sinh(\lambda)}{\lambda} \\ -\cos(\lambda) - \cosh(\lambda) & \sin(\lambda) - \sinh(\lambda) & L \cdot \frac{\cosh(\lambda)}{\lambda} + R \cdot \sinh(\lambda) \\ 2 \cdot \delta \cdot \lambda & -2 & L \cdot \varepsilon \cdot \lambda - \frac{3}{2} \delta \cdot L + R \end{bmatrix} \cdot \begin{matrix} A \\ C \\ \theta \end{matrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$D = R \cdot \theta - C$$

$$B = \frac{\theta}{\beta} - A \quad \beta = \frac{\lambda}{L} \quad \delta = \frac{R}{L} \quad \varepsilon = \frac{J}{\sigma \cdot L}$$

In the previously given expressions for kinetic energy, T , and potential energy, V , $y(x,t) = (R + x) \theta(t) + w(x,t)$ with its series representation was replaced. The result is:

$$T = \frac{1}{2} \cdot \sum_i \dot{q}_i^2 \cdot \left[J \cdot \left[\frac{d}{dx} \cdot \phi(0)_i \right]^2 + \int_0^L \sigma \cdot \phi(x)_i^2 dx \right]$$

$$V = \frac{1}{2} \cdot \sum_i q_i^2 \cdot \int E \cdot I \cdot \left[\frac{d^2}{dx^2} \cdot \phi(x)_i \right]^2 dx - u(t) \cdot \left[\sum_i \left[\frac{d}{dx} \cdot \phi(0)_i \right] \cdot q_i \right]$$

The modes were normalized so that the expression $\left[J \cdot \left[\frac{d}{dx} \cdot \phi(0)_i \right]^2 + \int_0^L \sigma \cdot \phi(x)_i^2 dx \right]$ equals one for each mode shape.

When this is done, the expressions for T and V reduce to the following:

$$T = \frac{1}{2} \cdot \sum_i \dot{q}_i^2 \quad V = \frac{1}{2} \cdot \sum_i q_i^2 \cdot \ddot{w}_i^2 - u(t) \cdot \sum_i q_i \cdot \left[\frac{d}{dx} \cdot \phi(0)_i \right]$$

The summation is over the range 0 to 3 where the value $i = 0$ corresponds to the rigid body mode for which the frequency of vibration is 0. Lagrange's equations were then applied, and a set of decoupled ordinary differential equations of the following form were obtained:

$$q_i + q_i \cdot \ddot{\omega}_i^2 = u(t) \cdot \frac{d}{dx} \cdot \phi(0)_i \quad i = 0, 1, 2, 3$$

These equations are easily put in state-space form and are shown in appendix E. The Bode plots of magnitude (dB) versus frequency (rad/sec) for disk angle motion and beam tip response and the beam tip response and disk angle response to a unit step torque are shown in figure 7. These results are slightly different from previous models as slightly different values of the parameters J,R,L and d were used to match actual experimental parameters.

Complete Model of Electromechanical System and Experimental Verification

The reduced-order model with system modes derived in the previous section was used as the model for the mechanical part of the system. The following equations of motion for the electrical part of the system must be added.

State variables: i_1 and i_2

$$\dot{i}_1 = \frac{1}{L_1} \cdot [-R_1 \cdot i_1 - K_b1 \cdot \theta + K_a1 \cdot e(t)_1]$$

$$\dot{i}_2 = \frac{1}{L_2} \cdot [-R_2 \cdot i_2 - K_b2 \cdot \theta + K_a2 \cdot e(t)_2]$$

The torque on the disk from the motors is given by:

$$\text{Torque from motors} = K_t1 \cdot i_1 + K_t2 \cdot i_2$$

R_1 = electrical resistance of motor 1

R_2 = electrical resistance of motor 2

L_1 = electrical inductance of motor 1

L_2 = electrical inductance of motor 2

K_a1 = amplifier gain for motor 1 (amplifier 1 was linear in the range $-1 < 0 < 1$ input volts)

K_a2 = amplifier gain for motor 2 (amplifier 2 was linear in the range $-1 < 0 < 1$ input volts)

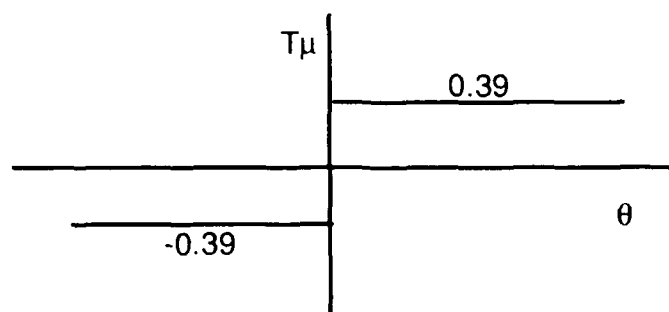
K_b1 = back emf constant for motor 1

K_b2 = back emf constant for motor 2

$e(t)_1$ = input voltage to motor 1

$e(t)_2$ = input voltage to motor 2

To this torque must be added the torque due to Coulomb friction, T_μ , which is represented by the following diagram:



The complete equations of motion for the electromechanical system are given in state-space form in appendix F along with a summary of all the experimentally determined physical parameters of the dynamic system.

The results of two open-loop simulations for the complete electromechanical system along with an example of the block diagram used in MATRIXx for the simulation are shown in figure 8 through 11.

In the first case, the input from both motors is a sine wave of magnitude 0.5 volts and frequency 8.5 Hz, which is the fundamental frequency of the mechanical system. Only the output, the disk angle, from this simulation is shown.

In the second case, the input from both motors is a triangular wave of magnitude 0.125 volts and frequency 0.5 Hz. Both the input from a motor and the output, the disk angle, are shown.

CONCLUSIONS

In recent years, the use of lightweight materials with distributed flexibility in advanced space applications and in the construction of robotic manipulators has led to a new and challenging problem in the combined area of modelling and control. Increasing interest has arisen to properly account for the inherent flexibility of these structures, the coupling between the rigid and flexible motions, and the interaction between the control system and the flexible structure itself.

In this report, the problem of modelling a flexible beam fixed to a rigid rotating disk has been analyzed in great detail. The equations of motion and boundary conditions developed by a means of Hamilton's Principle were considered. The effect of the radius and mass moment of inertia of the rigid disk on the fundamental frequency of the system was observed. Unconstrained and constrained mode expansions were obtained for the system. In the absence of controller dynamics, it was found that the constrained mode expansion yields very good results only if the flexible/rigid inertia ratio is small.

Finite element and lumped parameter models of the system were also derived. The motor dynamics were integrated into the structural model considering a specific laboratory setup. This experiment was also used to validate the model.

Multiple-link robot arms with motors at the joints can be considered to satisfy the condition of small flexible/rigid inertia ratio so that constrained expansions are expected to perform well. However, a question which requires further research effort is whether constrained mode expansions together with controller dynamics are suitable for currently projected space structures that have very large flexible/rigid inertia ratios.

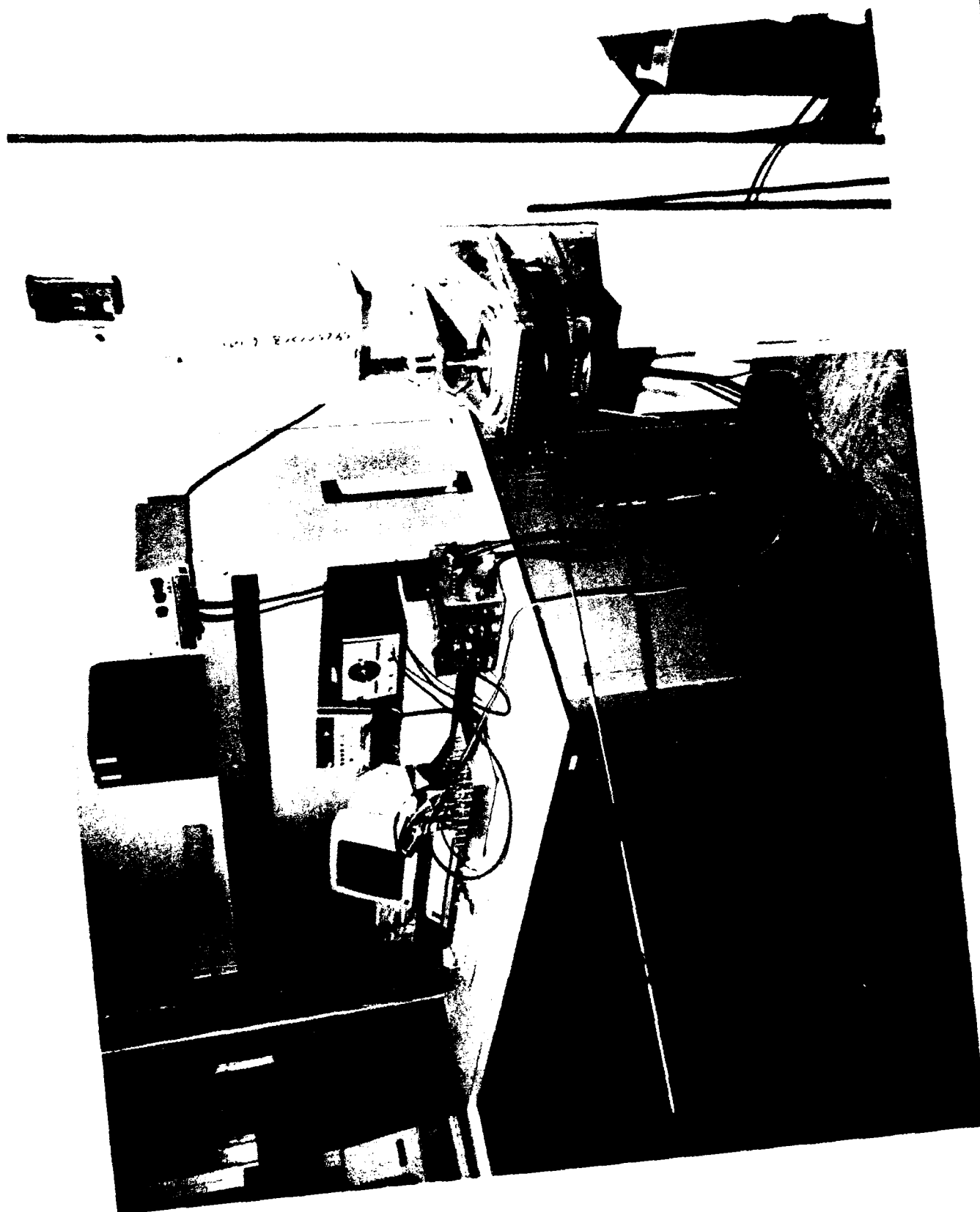


Figure 1. Experimental apparatus

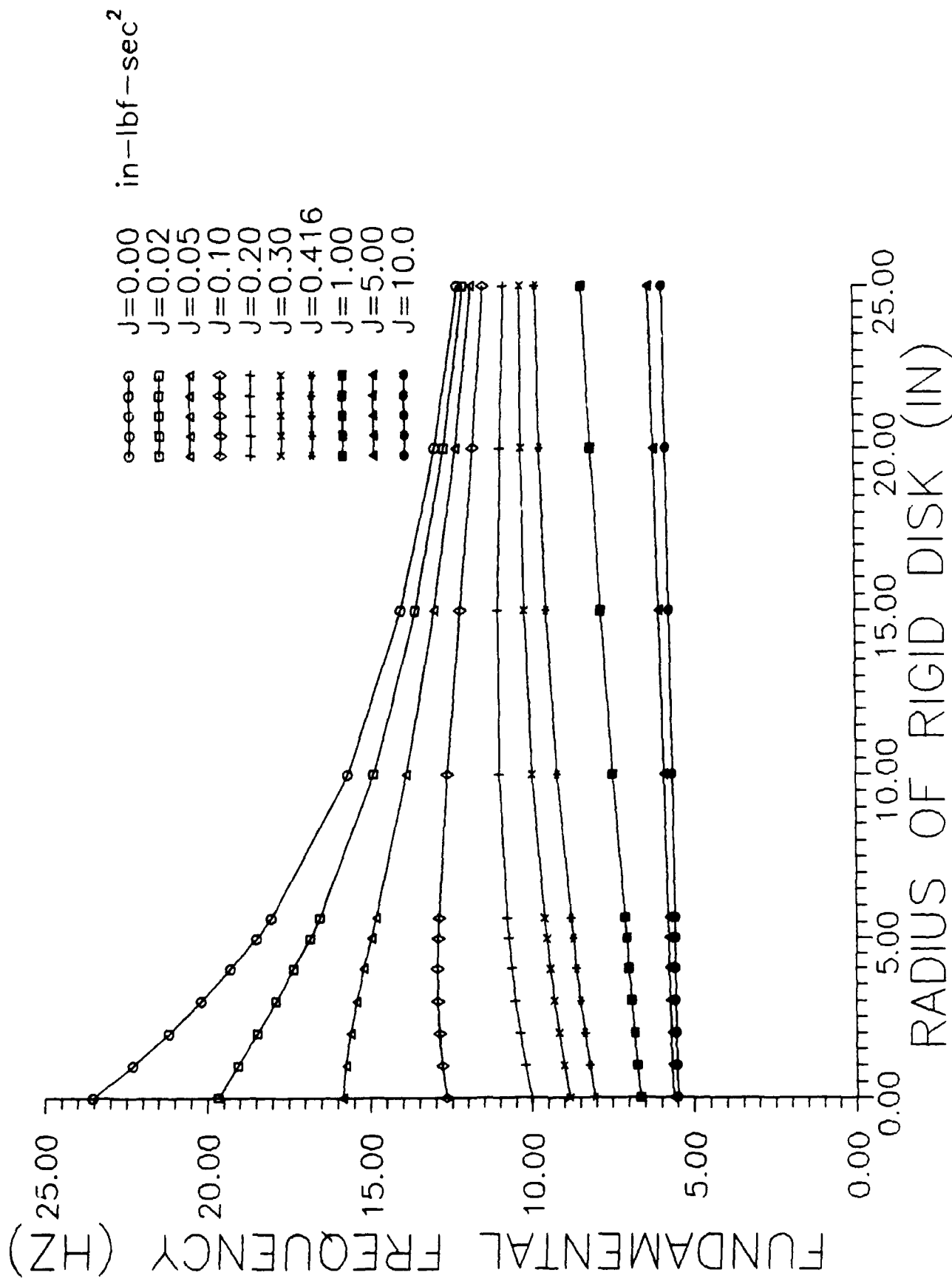


Figure 2. Dynamic analysis of a flexible beam attached to a rigid rotor with variable radius

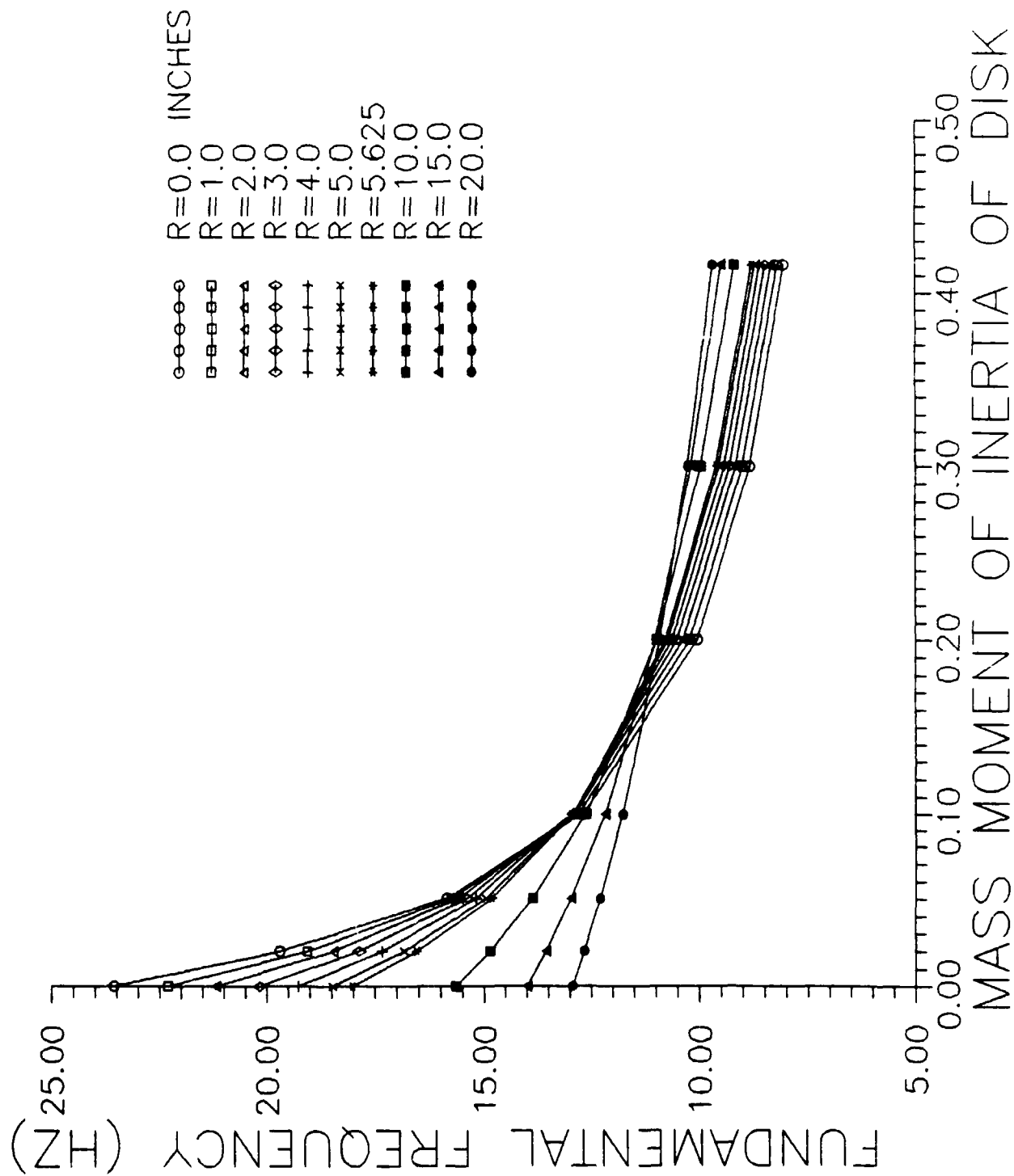
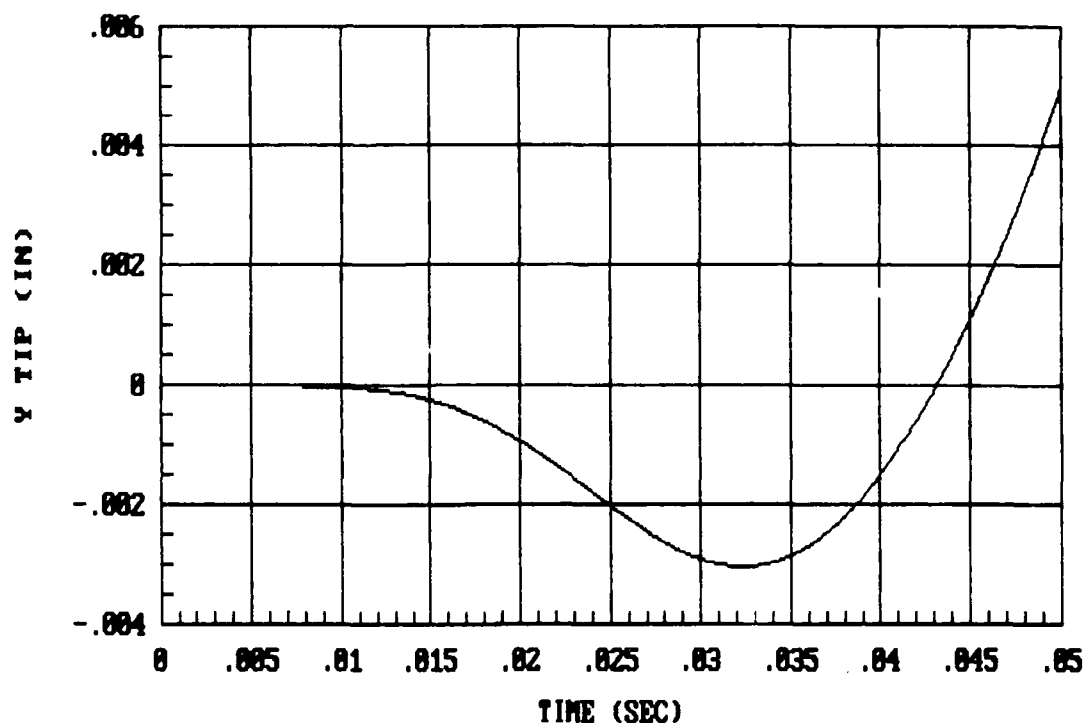
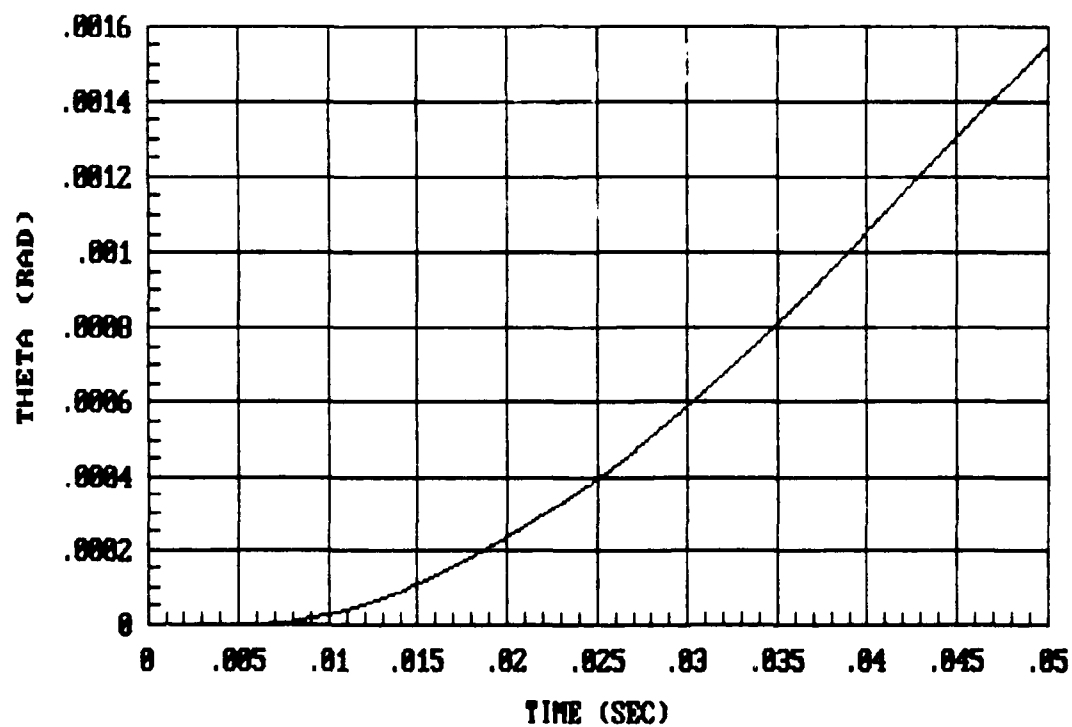


Figure 3. Dynamic analysis of a flexible beam attached to a rigid rotor with variable inertia



TIP RESPONSE TO UNIT STEP TORQUE AT $T=0.005$ SEC



DISK ANGLE RESPONSE TO UNIT STEP TORQUE AT $T=0.005$ SEC

Figure 4. Results from exact analysis

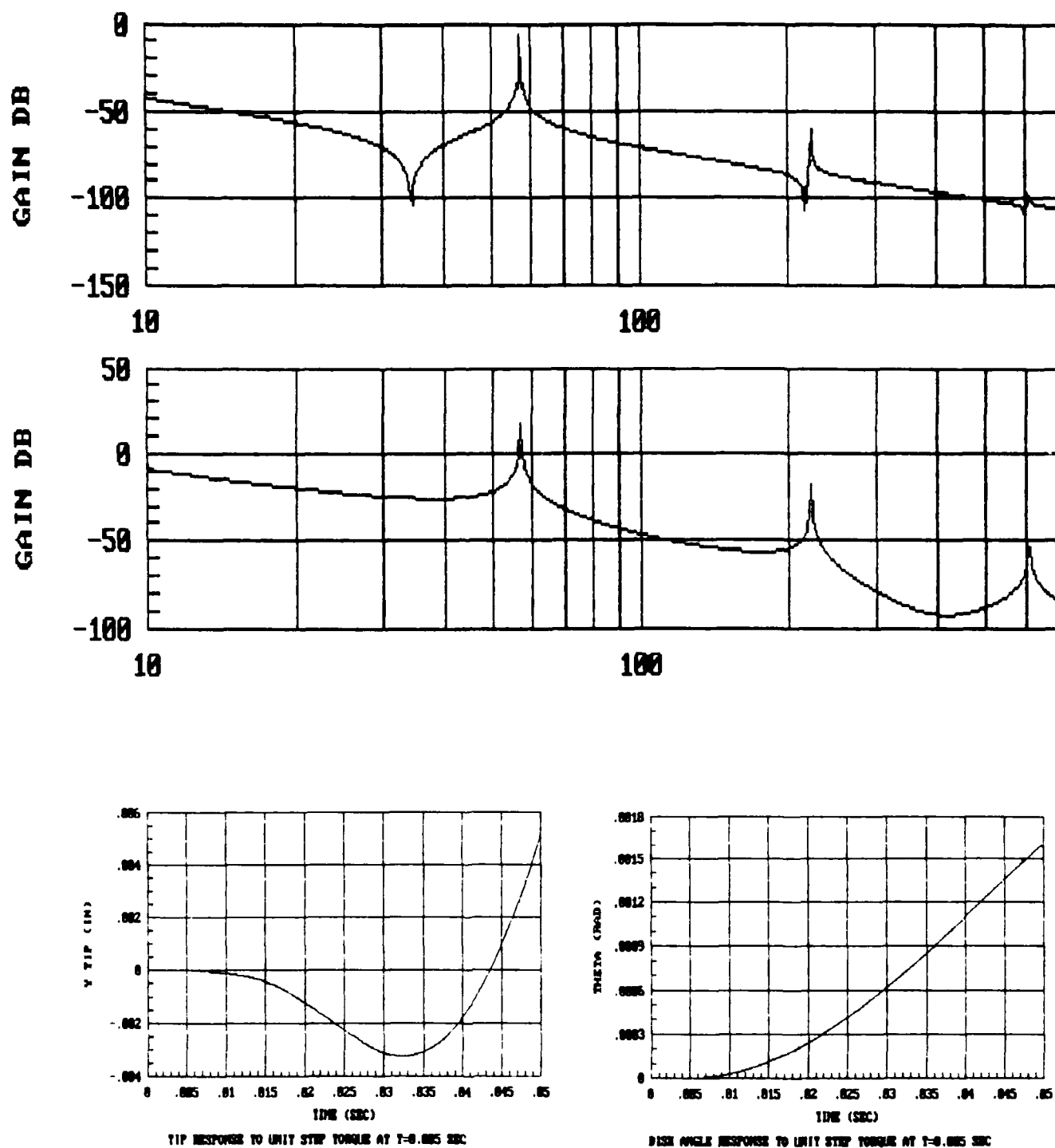


Figure 5. Results from reduced-order model with cantilevered modes

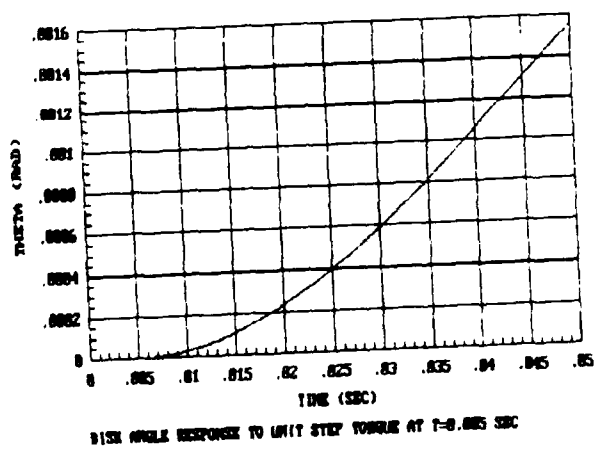
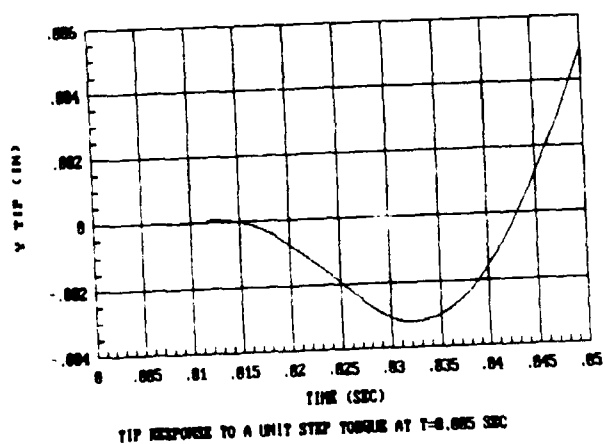
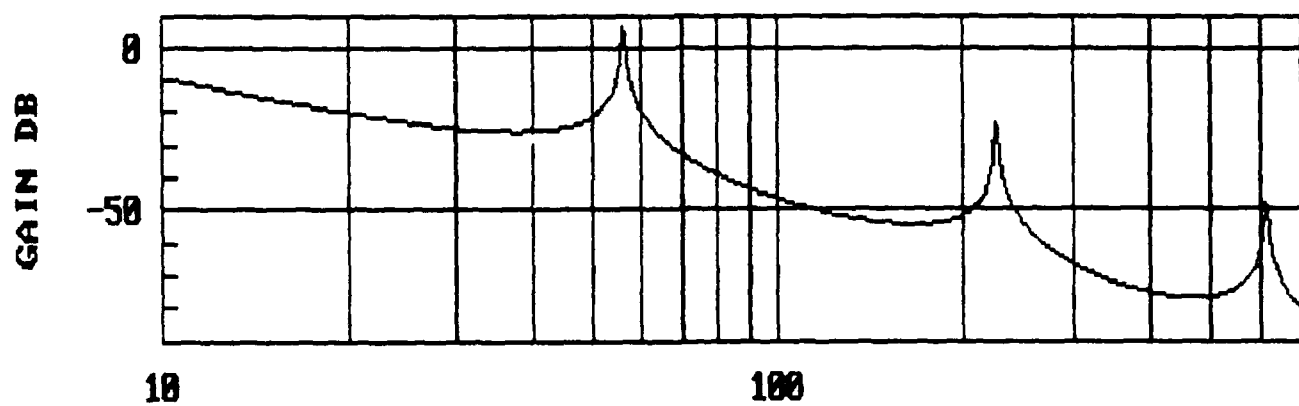
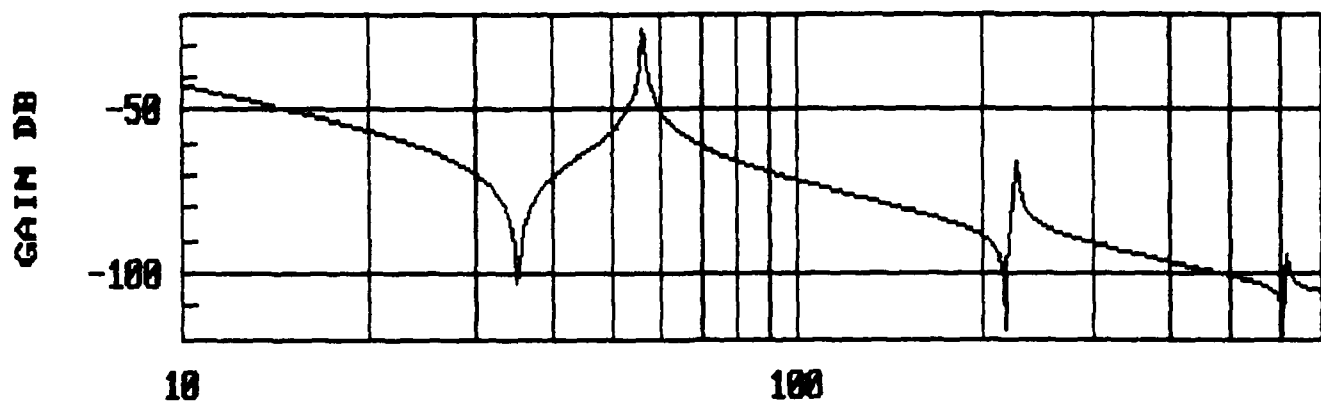
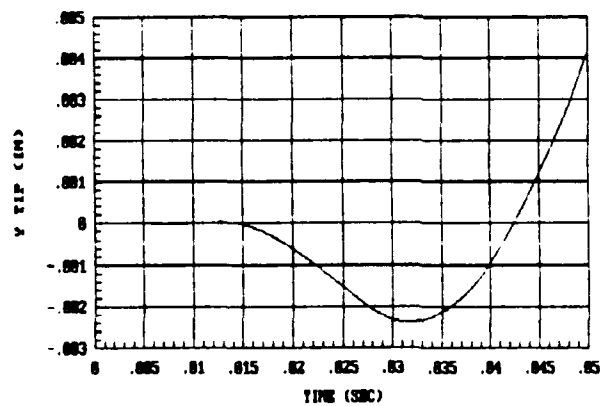
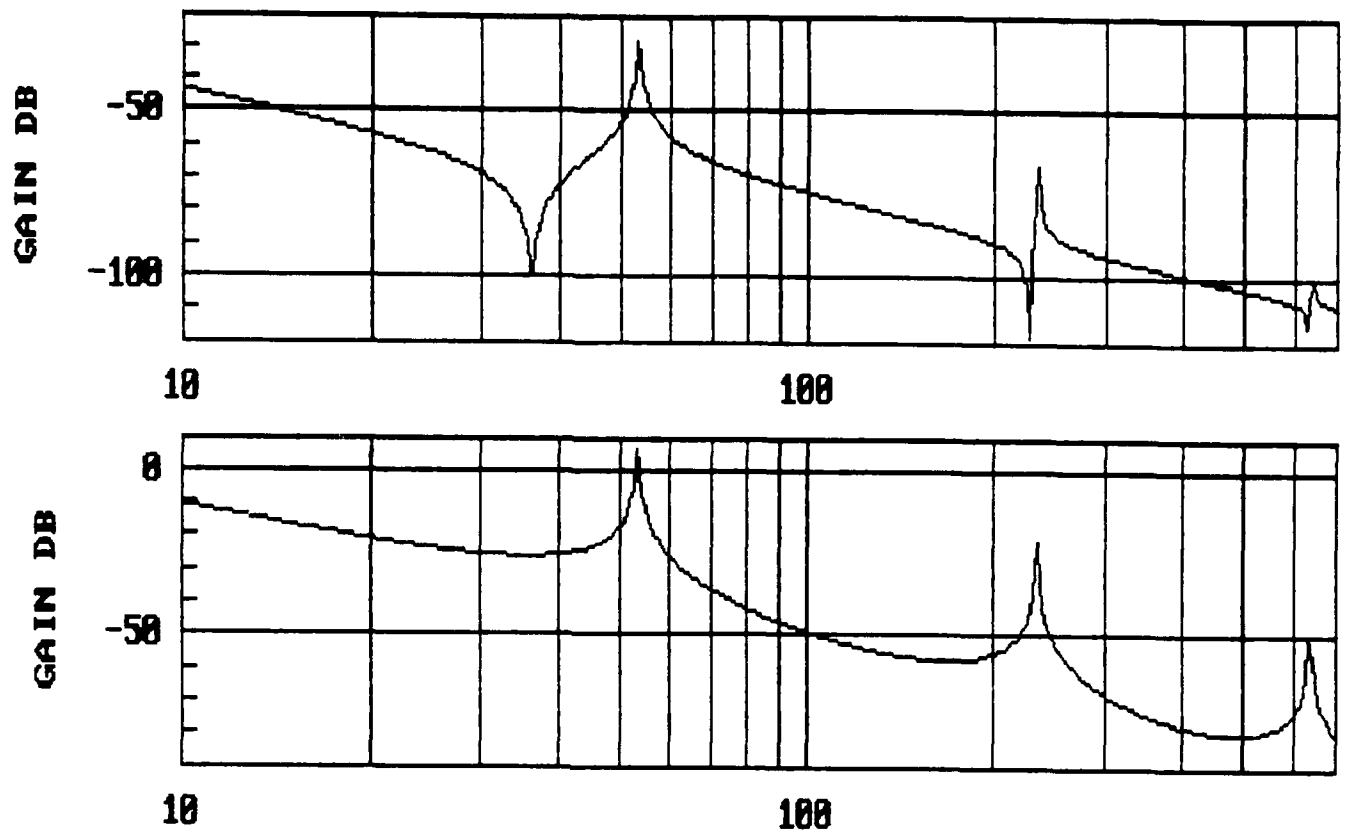
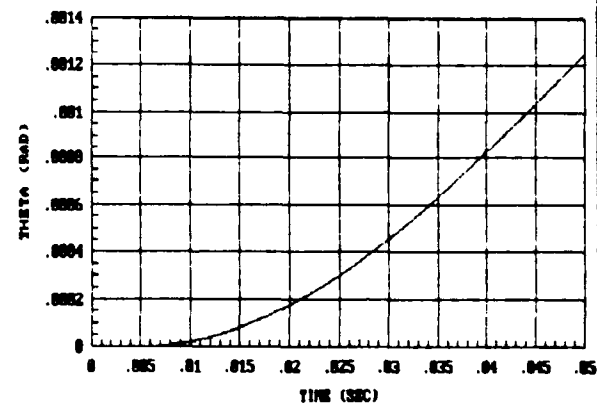


Figure 6. Results from finite element analysis model



TIP RESPONSE TO UNIT STEP TORQUE AT $t = 0.005$ SEC



DISK ANGLE RESPONSE TO UNIT STEP TORQUE AT $t = 0.005$ SEC

Figure 7. Results from reduced-order model with system modes

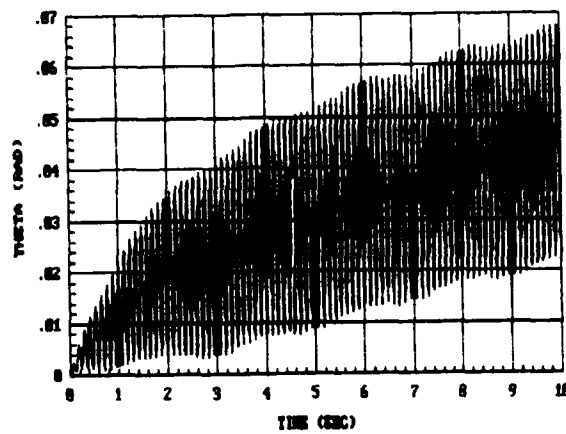


Figure 8. Open loop response to sine input

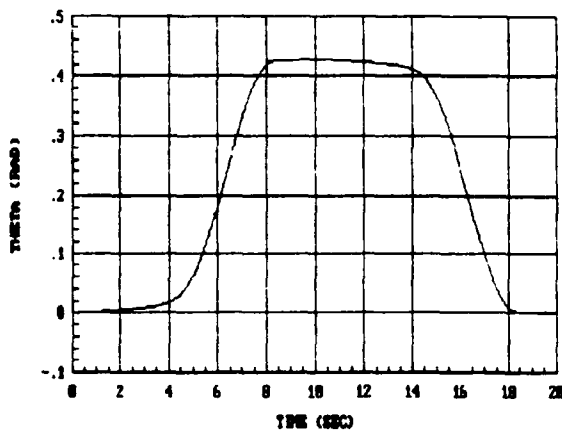


Figure 9. Open loop response to triangle input

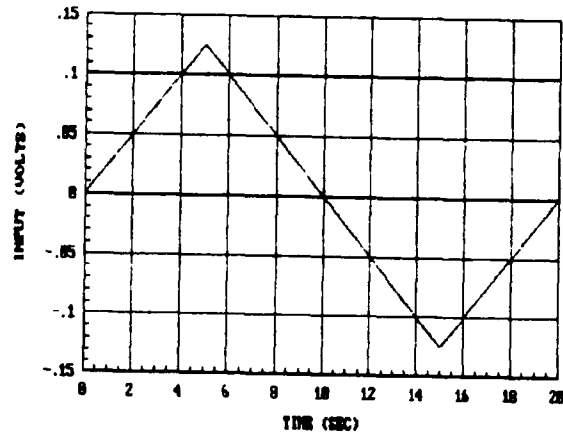


Figure 10. Triangle input

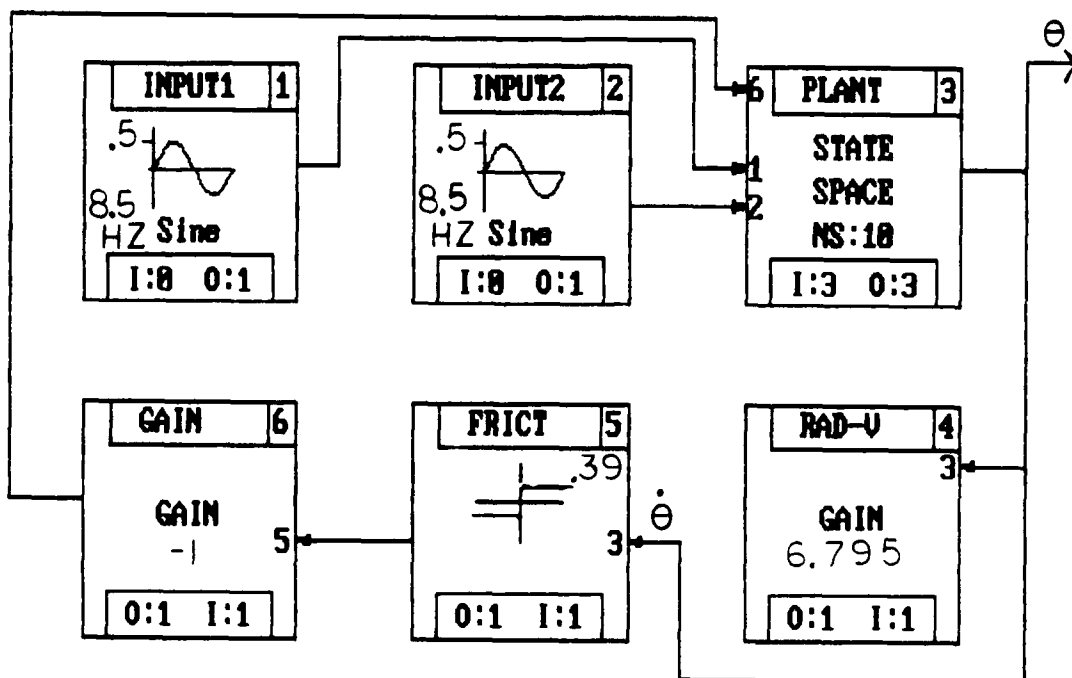


Figure 11. MATRIX_x model for simulation

APPENDIX A

RIGID DISK AND FLEXIBLE BEAM: EXACT OPEN-LOOP TRANSFER FUNCTIONS AND REDUCED-ORDER MODELS OF THE OPEN-LOOP TRANSFER FUNCTIONS

Parameter values (units: inch-lbf-sec)

$$E := 3.0 \cdot 10^7$$

$$L := 36.0$$

$$d := 0.25$$

$$\delta(R) := \frac{R}{L}$$

$$\rho := 0.000725$$

$$A := \pi \cdot \frac{d^4}{64}$$

$$I := \pi \cdot \frac{d^4}{64}$$

$$\sigma := \rho \cdot A$$

$$\varepsilon(J) := \frac{J}{\sigma \cdot L^3}$$

$$f(\lambda) := \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{E \cdot I}{\sigma \cdot L^4}} \cdot \lambda^2$$

$$\ddot{\omega}(\lambda) := 2 \cdot \pi \cdot f(\lambda)$$

$$R := 5.625$$

$$J := 0.416$$

$$N(\lambda) := 1 + \cos(\lambda) \cdot \cosh(\lambda)$$

$$NTIP(\lambda, R) := L \cdot [\sin(\lambda) + \sinh(\lambda)] + R \cdot \lambda \cdot (\cos(\lambda) + \cosh(\lambda))$$

$$D(\lambda, R, J) := \varepsilon(J) \cdot \lambda^3 \cdot [1 + \cos(\lambda) \cdot \cosh(\lambda)] + [1 + \lambda^2 \cdot \delta(R)^2] \cdot \sin(\lambda) \cdot \cosh(\lambda) \dots$$

$$+ 2 \cdot \lambda \cdot \delta(R) \cdot \sin(\lambda) \cdot \sinh(\lambda) + [\lambda^2 \cdot \delta(R)^2 - 1] \cdot \cos(\lambda) \cdot \sinh(\lambda)$$

$$TF(\lambda, R, J) := \frac{-L}{E \cdot I \cdot \lambda} \cdot \frac{N(\lambda)}{D(\lambda, R, J)}$$

$$TFTIP(\lambda, R, J) := \frac{-I}{E \cdot I \cdot \lambda^2} \cdot \frac{NTIP(\lambda, R)}{D(\lambda, R, J)}$$

$$\lambda := 1.8 \quad \text{root}[N(\lambda), \lambda] = 1.875$$

$$f(1.875) = 5.489$$

$$\ddot{\omega}(1.875) = 34.488$$

$$\lambda := 4.6 \quad \text{root}[N(\lambda), \lambda] = 4.694$$

$$f(4.694) = 34.401$$

$$\ddot{\omega}(4.694) = 216.149$$

$$\lambda := 7.8 \quad \text{root}[N(\lambda), \lambda] = 7.855$$

$$f(7.855) = 96.334$$

$$\ddot{\omega}(7.855) = 605.284$$

$$\lambda := 10.9 \quad \text{root}[N(\lambda), \lambda] = 10.996$$

$$f(10.996) = 188.78$$

$$\ddot{\omega}(10.996) = 1.186 \cdot 10^3$$

$$\lambda := 2.3 \quad \text{root}[D(\lambda, R, J), \lambda] = 2.3993$$

$$f(2.393) = 8.941$$

$$\ddot{\omega}(2.393) = 56.176$$

$\lambda := 4.8$	$\text{root}[D(\lambda, R, J), \lambda] = 4.807$	$f(4.807) = 36.077$
		$\ddot{\omega}(4.807) = 226.681$
$\lambda := 7.8$	$\text{root}[D(\lambda, R, J), \lambda] = 7.895$	$f(7.895) = 97.318$
		$\ddot{\omega}(7.895) = 611.464$
$\lambda := 10.9$	$\text{root}[D(\lambda, R, J), \lambda] = 11.018$	$f(11.018) = 189.536$
		$\ddot{\omega}(11.018) = 1.191 \cdot 10^3$
$\lambda := 2.0 + 2.0 \cdot i$	$\text{root}[NTIP(\lambda, R), \lambda] = 2.12 + 2.12i$	$ 2.12 + 2.12 \cdot i = 2.998$
$\lambda := 5.0 + 5.0 \cdot i$	$\text{root}[NTIP(\lambda, R), \lambda] = 5.081 + 5.081i$	$ 5.081 + 5.081 \cdot i = 7.186$
$\lambda := 8.0 + 8.0 \cdot i$	$\text{root}[NTIP(\lambda, R), \lambda] = 8.129 + 8.129i$	$ 8.129 + 8.129 \cdot i = 11.496$
$\lambda := 11.0 + 11.0 \cdot i$	$\text{root}[NTIP(\lambda, R), \lambda] = 11.214 + 11.214i$	$ 11.214 + 11.214 \cdot i = 15.859$
$\lambda := 0$		$\ddot{\omega}(2.998) = 88.172$
$N(\lambda) = 2$		$\ddot{\omega}(7.186) = 506.572$
		$\ddot{\omega}(11.496) = 1.296 \cdot 10^3$
$N1(\lambda) := \frac{d}{d\lambda} N(\lambda)$	$N1(\lambda) = 0$	
$N2(\lambda) := \frac{d}{d\lambda} N1(\lambda)$	$N2(\lambda) = -1.91 \cdot 10^{12}$	$\ddot{\omega}(15.859) = 2.467 \cdot 10^3$
$N3(\lambda) := \frac{d}{d\lambda} N2(\lambda)$	$N3(\lambda) = 0$	
$D(\lambda, R, J) = 0$		
$D1(\lambda, R, J) := \frac{d}{d\lambda} D(\lambda, R, J)$		$D1(\lambda, R, J) = 1.563 \cdot 10^{11}$
$D2(\lambda, R, J) := \frac{d}{d\lambda} D1(\lambda, R, J)$		$D2(\lambda, R, J) = 0$

$$D3(\lambda, R, J) := \frac{d}{d\lambda} D2(\lambda, R, J)$$

$$D3(\lambda, R, J) = 9.174$$

$$NTIP(\lambda, R) = 0$$

$$NTIP1(\lambda, R) := \frac{d}{d\lambda} NTIP(\lambda, R)$$

$$NTIP1(\lambda, R) = 83.25$$

$$NTIP2(\lambda, R) := \frac{d}{d\lambda} NTIP1(\lambda, R)$$

$$NTIP2(\lambda, R) = 0$$

$$NTIP3(\lambda, R) := \frac{d}{d\lambda} NTIP2(\lambda, R)$$

$$NTIP3(\lambda, R) = -4.734 \cdot 10^{-11}$$

$$NN(\lambda) := \frac{4 \cdot L^2}{E \cdot I \cdot \lambda^2} \cdot \left[1 - \frac{\lambda^4}{1.874^4} \right] \cdot \left[1 - \frac{\lambda^4}{4.694^4} \right] \cdot \left[1 - \frac{\lambda^4}{7.855^4} \right] \cdot \left[1 - \frac{\lambda^4}{10.996^4} \right]$$

$$DD(\lambda) := \frac{-L \cdot \lambda^2}{0.327011} \cdot \left[1 - \frac{\lambda^4}{2.393^4} \right] \cdot \left[1 - \frac{\lambda^4}{4.807^4} \right] \cdot \left[1 - \frac{\lambda^4}{7.895^4} \right] \cdot \left[1 - \frac{\lambda^4}{11.018^4} \right]$$

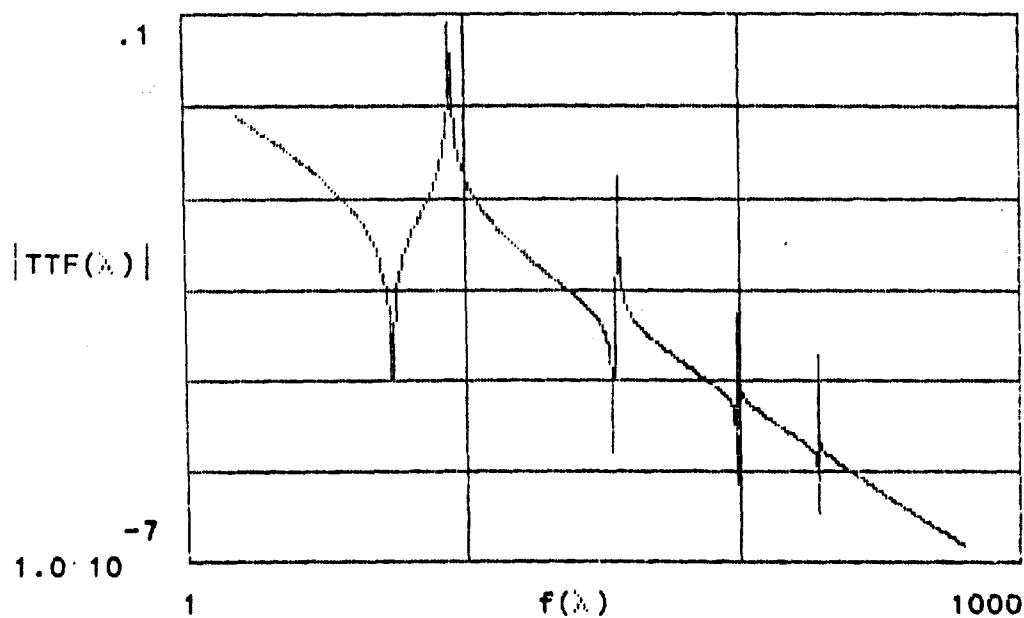
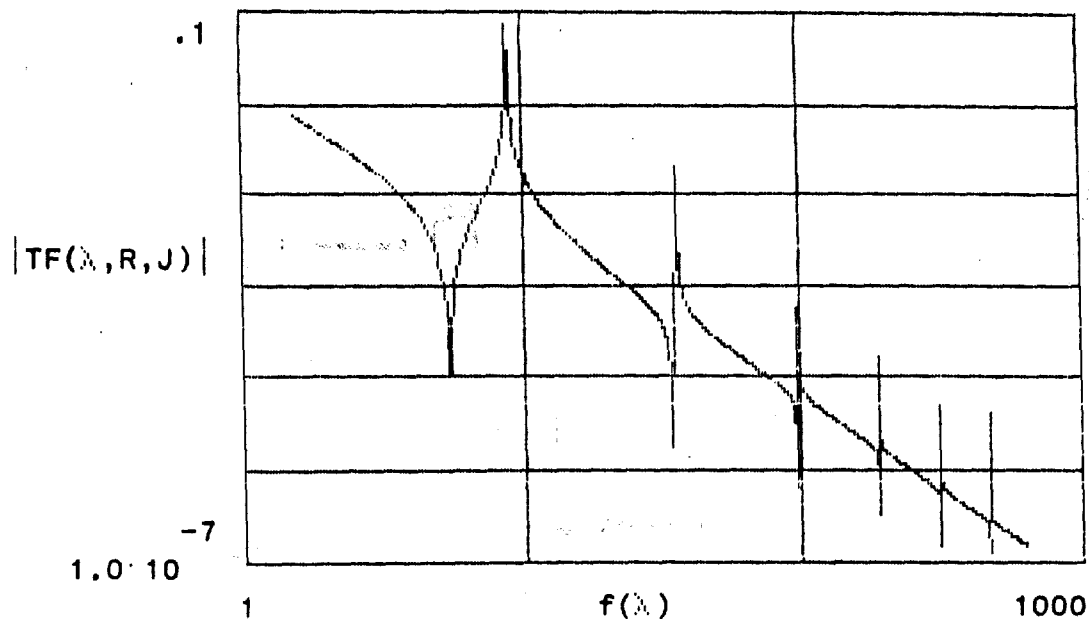
$$NNTIP(\lambda) := \frac{166.5 \cdot L^2}{E \cdot I \cdot \lambda^2} \cdot \left[1 + \frac{\lambda^4}{2.998^4} \right] \cdot \left[1 + \frac{\lambda^4}{7.186^4} \right] \cdot \left[1 + \frac{\lambda^4}{11.496^4} \right] \cdot \left[1 + \frac{\lambda^4}{15.859^4} \right]$$

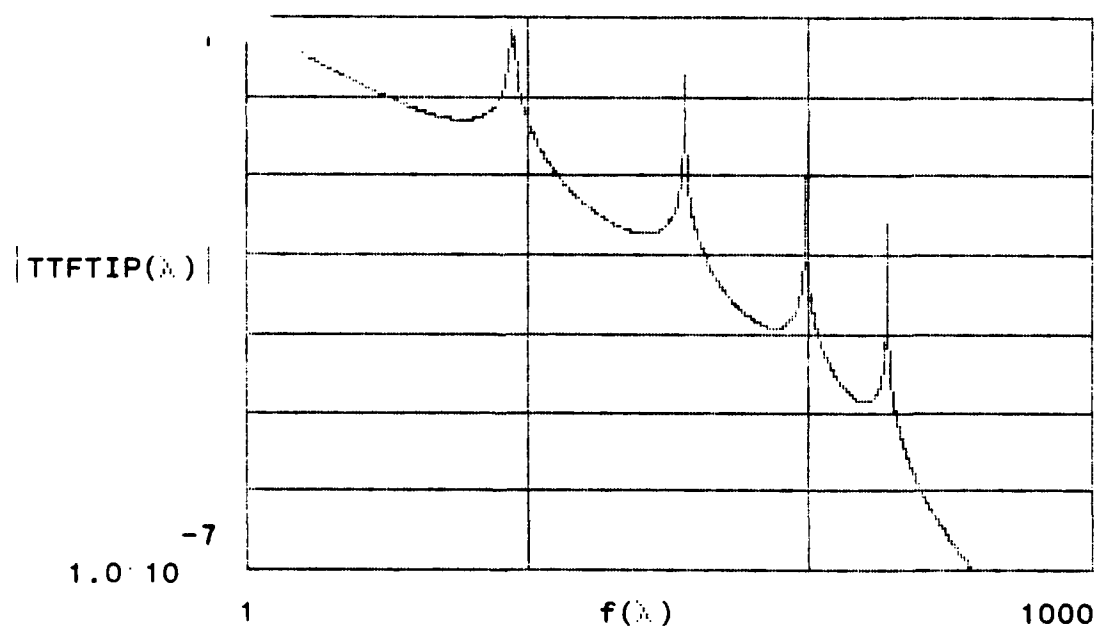
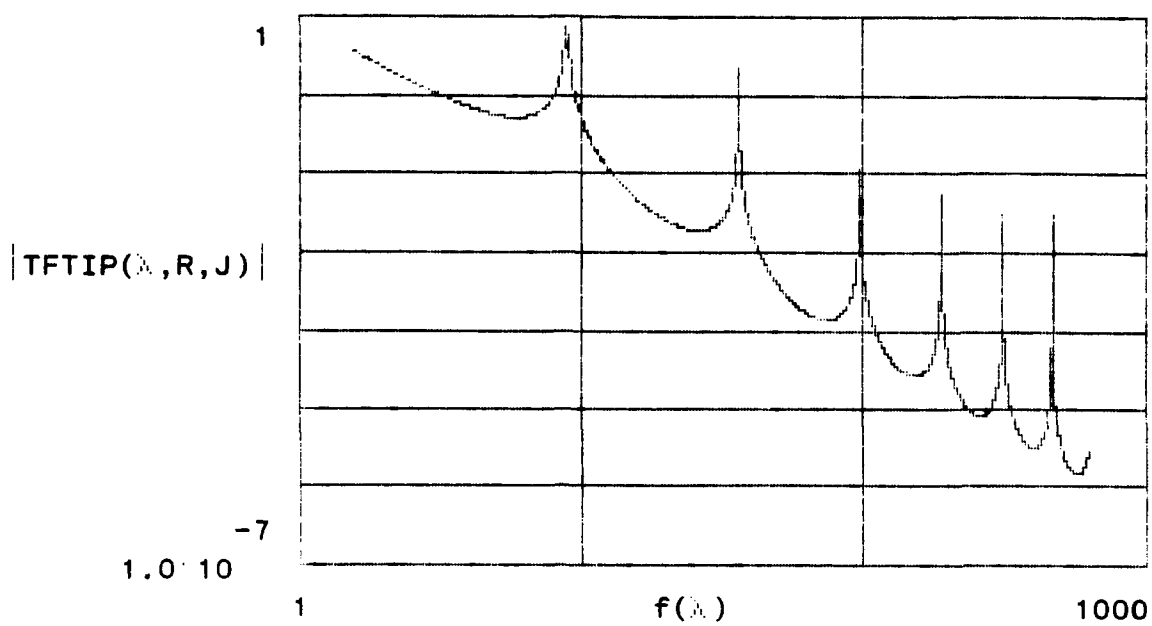
$$TTF(\lambda) := \frac{NN(\lambda)}{DD(\lambda)}$$

$$TTFTIP(\lambda) := \frac{NNTIP(\lambda)}{DD(\lambda)}$$

Bode Plot Comparison: Exact
versus Reduced Order

$$\lambda := 1, 1.01 \dots 20$$





APPENDIX B

REDUCED-ORDER MODEL OF RIGID DISK/FLEXIBLE
BEAM SYSTEM WITH CANTILEVERED MODES

Parameter values (units: inch-lbf-sec)

$$E := 3.0 \cdot 10^7$$

$$L := 36.0$$

$$d := 0.25$$

$$\delta(r) := \frac{R}{L}$$

$$\rho := 0.000725$$

$$I := \pi \cdot \frac{d^4}{64}$$

$$A := \pi \cdot \frac{d^2}{4}$$

$$\varepsilon(J) := \frac{J}{\sigma \cdot L^3}$$

$$F(\lambda) := \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{E \cdot I}{\sigma \cdot L^4}} \lambda^2$$

$$\omega(\lambda) := 2 \cdot \pi \cdot f(\lambda)$$

$$R := 5.625$$

$$J := 0.416$$

Mode shape of a cantilever beam: $\phi(x)$

Origin = 1

$$\lambda_1 := 1.8751$$

$$\lambda_2 := 4.69409$$

$$\lambda_3 := 7.85476$$

$$F(\lambda) := \frac{\cosh(\lambda) + \cos(\lambda)}{\sinh(\lambda) + \sin(\lambda)}$$

$$F(\lambda_1) = 0.734096$$

$$F(\lambda_2) = 1.018467$$

$$F(\lambda_3) = 0.999224$$

$$\sigma(x, \lambda) := -L \cdot \left[\cos\left[\frac{\lambda}{L} \cdot x\right] - \cosh\left[\frac{\lambda}{L} \cdot x\right] - F(\lambda) \cdot \left[\sin\left[\frac{\lambda}{L} \cdot x\right] - \sinh\left[\frac{\lambda}{L} \cdot x\right] \right] \right]$$

$$\phi(L, \lambda_1) = 71.999785$$

$$\phi(L, \lambda_2) = -71.999917$$

$$\phi(L, \lambda_3) = 72.000184$$

$$F1 := \sigma \cdot L \cdot \left[R^2 + R \cdot L + \frac{L^2}{3} \right]$$

$$F2(\lambda) := 2 \cdot R \cdot \int_0^L \phi(x, \lambda) \cdot \sigma \, dx$$

$$F3(\lambda) := 2 \cdot \int_0^L x \cdot \phi(x, \lambda) \cdot \sigma \, dx$$

$$F4(\lambda) := \int_0^L \sigma \cdot [\phi(x, \lambda)]^2 \, dx$$

$$F5(\lambda) := \int_0^L E \cdot I \cdot \left[\frac{d}{dx} \left[\frac{d}{dx} \phi(x, \lambda) \right] \right]^2 \, dx$$

$$F1 = 0.853447$$

$$F21 := F2(\lambda_1)$$

$$F21 = 0.406276$$

$$F31 := F3(\lambda_1)$$

$$F41 := F4(\lambda_1)$$

$$F51 := F5(\lambda_1)$$

$$F22 := F2(\lambda_2)$$

$$F32 := F3(\lambda_2)$$

$$F42 := F4(\lambda_2)$$

$$F52 := F5(\lambda_2)$$

$$F23 := F2(\lambda_3)$$

$$F33 := F3(\lambda_3)$$

$$F43 := F4(\lambda_3)$$

$$F53 := F5(\lambda_3)$$

$$F31 = 1.888962$$

$$F41 = 1.660399$$

$$F51 = 1.975365 \cdot 10^3$$

$$F22 = 0.225161$$

$$F32 = 0.30142$$

$$F42 = 1.660409$$

$$F52 = 7.758082 \cdot 10^4$$

$$F23 = 0.132019$$

$$F33 = 0.107644$$

$$F43 = 1.660408$$

$$F53 = 6.082459 \cdot 10^5$$

State variables: $x_1 = \theta$

$x_5 = q_2$

$x_2 = \dot{\theta}$

$x_6 = \dot{q}_2$

$x_3 = q_1$

$x_7 = q_3$

$x_4 = \dot{q}_1$

$x_8 = \dot{q}_3$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.206102 \cdot 10^3 & 0 & 2.88881 \cdot 10^4 & 0 & 1.030814 \cdot 10^5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3.405655 \cdot 10^3 & 0 & -1.99666 \cdot 10^4 & 0 & -7.124681 \cdot 10^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -254.195034 & 0 & -4.901431 \cdot 10^4 & 0 & -8172785 \cdot 10^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -115.692072 & 0 & -1.042426 \cdot 10^3 & 0 & -3.700429 \cdot 10^5 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 2.348253 \\ 0 \\ -1.623043 \\ 0 \\ -0.186181 \\ 0 \\ -0.084737 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 41.625 & 0 & 72 & 0 & -72 & 0 & 72 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

APPENDIX C

FINITE ELEMENT ANALYSIS OF RIGID DISK/FLEXIBLE BEAM SYSTEM

Using six beam elements to represent the flexible beam offset from the center of rotation of the disk and a lumped inertia to represent the rigid disk, the mass matrix, [M], and stiffness matrix, [K], are:

$$\begin{bmatrix} 4.201 \cdot 10^4 & 14.94 & -8.729 & 3.662 & -1.321 & 0.4059 & -0.1055 \\ & 17.52 & 2.644 & -1.361 & 0.5450 & -0.1762 & 0.04927 \\ & & 18.52 & 2.207 & -1.185 & 0.4371 & -0.1429 \\ & & & 18.70 & 2.099 & -1.009 & 0.4293 \\ & & & & 18.88 & 1.662 & -1.227 \\ & & & & & 19.88 & 3.345 \\ & & & & & & 5.992 \end{bmatrix} \cdot 10^5 = [M]$$

$$\begin{bmatrix} 1.722 \cdot 10^2 & -24.85 & 9.041 & -2.413 & 0.6413 & -0.1599 & 0.02659 \\ & 4.997 & -3.159 & 1.271 & -0.3376 & 0.08415 & -0.01400 \\ & & 3.895 & -2.863 & 1.186 & -0.2957 & 0.04918 \\ & & & 3.811 & -2.821 & 1.102 & -0.1833 \\ & & & & 3.727 & -2.525 & 0.6863 \\ & & & & & 2.625 & -0.9692 \\ & & & & & & 0.4275 \end{bmatrix} \cdot 10^2 = [K]$$

$$\begin{bmatrix} \theta \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = [q] \text{ nodal displacement vector} \quad \begin{bmatrix} \text{Torque} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [Q] \text{ external force vector}$$

Solution to the eigenvalue problem: $[K] [u] = [\omega^2] [M] [u]$

$$\begin{aligned} \ddot{\omega}_0^2 &= 0 & \ddot{\omega}_1^2 &= 3.1938 \cdot 10^3 & \ddot{\omega}_2^2 &= 5.1292 \cdot 10^4 & \ddot{\omega}_3^2 &= 3.7616 \cdot 10^5 \\ & & \ddot{\omega}_4^2 &= 1.4584 \cdot 10^6 & \ddot{\omega}_5^2 &= 4.1546 \cdot 10^6 & \ddot{\omega}_6^2 &= 9.0099 \cdot 10^6 \end{aligned}$$

$$\begin{bmatrix} 2.469 & -2.535 & 0.8549 & -0.3914 & 0.2364 & -0.1620 & 0.1389 \\ 28.68 & -20.82 & -15.69 & 46.52 & -67.04 & 73.36 & -77.37 \\ 43.34 & -12.27 & -54.78 & 72.19 & -19.63 & -48.45 & 89.98 \\ 57.74 & 7.786 & -71.04 & 3.162 & 69.07 & -0.9024 & -94.63 \\ 71.91 & 35.43 & -44.11 & -63.90 & -16.45 & 50.69 & 87.39 \\ 85.97 & 67.07 & 20.09 & -21.95 & -52.03 & -64.88 & -66.79 \\ 100 & 100 & 100 & 100 & 100 & 100 & 100 \end{bmatrix} \cdot 10^{-2} = [u]$$

Normal modes must be normalized so that: $[u]^T [M] [u] = [I]$
 $[u]^T [K] [u] = [\omega^2]$

The result is:

$$\begin{bmatrix} 0.8985 & -1.1314 & 0.4669 & -0.2201 & 0.1388 & -0.1002 & 0.0660 \\ 10.4357 & -9.2940 & -8.5689 & 26.1614 & -39.3613 & 45.3776 & -36.7462 \\ 15.7682 & -5.4752 & -29.9182 & 40.5986 & -11.5230 & -29.9682 & 42.7346 \\ 21.0086 & 3.4755 & -38.7995 & 1.7785 & 40.5556 & -0.5582 & -44.9412 \\ 26.1672 & 15.8138 & -24.0894 & -35.9363 & -9.6564 & 31.3545 & 41.5060 \\ 31.2821 & 29.9402 & 10.9742 & -12.3428 & -30.5504 & -40.1319 & -31.7223 \\ 36.3838 & 44.6376 & 54.6172 & 56.2393 & 58.7158 & 61.8603 & 47.4935 \end{bmatrix} = [u]$$

The equations of motion become: $[\ddot{\beta}] + [\omega^2] [\beta] = [u]^T [Q]$

$$= \begin{bmatrix} 0.8985 \\ -1.1314 \\ 0.4669 \\ -0.2201 \\ 0.1388 \\ -0.1002 \\ 0.0660 \end{bmatrix} \quad [\text{Torque}]$$

The output equations are: $[q] = [u] [\beta]$

These equations are uncoupled and it is straightforward to put them in state variable form.

APPENDIX D

REDUCED-ORDER MODEL OF RIGID DISK - FLEXIBLE BEAM
WITH SYSTEM MODES

NOTE: Parameter values for L, d, R, and J are different here than in previous simulations. These are the actual parameters for the apparatus on which experiments were performed.

$$E := 3.0 \cdot 10^7$$

$$L := 35.125$$

$$d := 0.249$$

$$\delta(R) := \frac{R}{L}$$

$$\rho := 0.000725$$

$$AREA := \pi \cdot \frac{d^2}{4}$$

$$I := \pi \cdot \frac{d^4}{64}$$

$$\sigma := \rho \cdot AREA$$

$$\varepsilon(J) := \frac{J}{\sigma \cdot L^3}$$

$$F(\lambda) := \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{E \cdot I}{\sigma \cdot L^4}} \cdot \lambda^2$$

$$\ddot{\omega}(\lambda) := 2 \cdot \pi \cdot f(\lambda)$$

$$J := 0.57301$$

$$D1(\lambda, R, J) := \varepsilon(J) \cdot \lambda^3 \cdot [1 + \cos(\lambda) \cdot \cosh(\lambda)] + [1 + \lambda^2 \cdot \delta(R)^2] \cdot \sin(\lambda) \cdot \cosh(\lambda) \dots$$

$$+ 2 \cdot \lambda \cdot \delta(R) \cdot \sin(\lambda) \cdot \sinh(\lambda) + [\lambda^2 \cdot \delta(R)^2 - 1] \cdot \cos(\lambda) \cdot \sinh(\lambda)$$

$$D(\lambda, R, J) := \frac{-2 \cdot L}{\lambda} \cdot D1(\lambda, R, J)$$

$$\lambda := 2.3$$

$$\lambda 1 := \text{root}[d(\lambda, R, J), \lambda]$$

$$\lambda 1 = 2.281132052$$

$$f(\lambda 1) = 8.500007313$$

$$\ddot{\omega}(\lambda 1) = 53.407121063$$

$$\lambda := 4.8$$

$$\lambda 2 := \text{root}[D(\lambda, R, J), \lambda]$$

$$\lambda 2 = 4.777449767$$

$$f(\lambda 2) = 37.282987411$$

$$\ddot{\omega}(\lambda 2) = 234.255918709$$

$$\lambda := 7.8$$

$$\lambda 3 := \text{root}[D(\lambda, R, J), \lambda]$$

$$\lambda 3 = 7.884939741$$

$$f(\lambda 3) = 101.558248674$$

$$\ddot{\omega}(\lambda 3) = 638.109295891$$

$$\lambda := 10.9$$

$$\lambda 4 := \text{root}[d(\lambda, R, J), \lambda]$$

$$\lambda 4 = 11.012274831$$

$$f(\lambda 4) = 198.094550571$$

$$\ddot{\omega}(\lambda 4) = 1.24466477 \cdot 10^3$$

$$X11(\lambda) := -\sin(\lambda) - \sinh(\lambda)$$

$$X21(\lambda) := -\cos(\lambda) - \cosh(\lambda)$$

$$X12(\lambda) := -\cos(\lambda) - \cosh(\lambda)$$

$$X22(\lambda) := \sin(\lambda) - \sinh(\lambda)$$

$$X13(\lambda) := R \cdot \cosh(\lambda) + \frac{L \cdot \sinh(\lambda)}{\lambda}$$

$$X23(\lambda) := \frac{L \cdot \cosh(\lambda)}{\lambda} + R \cdot \sinh(\lambda)$$

$$X_{31}(\lambda, R) := 2 \cdot \delta(R) \cdot \lambda$$

$$X_{32} := -2$$

$$X_{33}(\lambda, J, R) := \frac{L \cdot \varepsilon(J) \cdot \lambda^3}{\lambda} - L \cdot \delta(R) + R$$

$$X(\lambda, R, J) := \begin{bmatrix} X_{11}(\lambda) & X_{12}(\lambda) & X_{13}(\lambda) \\ X_{21}(\lambda) & X_{22}(\lambda) & X_{23}(\lambda) \\ X_{31}(\lambda, R) & X_{32} & X_{33}(\lambda, J, R) \end{bmatrix}$$

$$\lambda := \lambda_1$$

$$\lambda = 2.281132052$$

$$X(\lambda, R, J) = \begin{bmatrix} -5.600935951 & -4.2928731 & 104.548427761 \\ -4.2928731 & -4.084649959 & 105.502227379 \\ 0.787437043 & -2 & 68.454784401 \end{bmatrix}$$

$$|X(\lambda, R, J)| = 1.370693935 \cdot 10^{-5}$$

$$\lambda := \lambda_2$$

$$\lambda = 4.777449767$$

$$x(\lambda, R, J) = \begin{bmatrix} -58.398403556 & -59.469720184 & 796.837320401 \\ -59.469720184 & -60.394172143 & 796.848176783 \\ 1.649155258 & -2 & 300.258431662 \end{bmatrix}$$

$$|x(\lambda, R, J)| = -6.553719071 \cdot 10^{-8}$$

$$\lambda := \lambda_3$$

$$\lambda = 7.884939741$$

$$X(\lambda, R, J) = \begin{bmatrix} -1.329481791 \cdot 10^3 & -1.328451693 \cdot 10^3 & 1.397190909 \cdot 10^4 \\ -1.328451693 \cdot 10^3 & -1.327482749 \cdot 10^3 & 1.397190849 \cdot 10^4 \\ 2.721847526 & -2 & 817.899062996 \end{bmatrix}$$

$$|X(\lambda, R, J)| = -4.009991272 \cdot 10^{-7}$$

The solution to the set of homogeneous equation with $\theta_i = 1$ is given by the following:

ORIGIN $\equiv 1$

$$i := 1, 2, 3$$

$$\lambda := \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

$$C := \begin{bmatrix} 31.938591 \\ 87.720856 \\ 179.353947 \end{bmatrix}$$

$$A := \begin{bmatrix} -5.813294 \\ -75.685245 \\ -168.705692 \end{bmatrix}$$

$$B := \begin{bmatrix} 21.211351 \\ 83.037494 \\ 173.160387 \end{bmatrix}$$

$$D := \begin{bmatrix} -25.876091 \\ -81.658356 \\ -173.291447 \end{bmatrix}$$

The modes of vibration are:

$$\phi_1(x) := A_1 \cdot \sin\left[\frac{\lambda_1}{L} \cdot x\right] + B_1 \cdot \sinh\left[\frac{\lambda_1}{L} \cdot x\right] + C_1 \cdot \cos\left[\frac{\lambda_1}{L} \cdot x\right] + D_1 \cdot \cosh\left[\frac{\lambda_1}{L} \cdot x\right]$$

$$\phi_2(x) := A_2 \cdot \sin\left[\frac{\lambda_2}{L} \cdot x\right] + B_2 \cdot \sinh\left[\frac{\lambda_2}{L} \cdot x\right] + C_2 \cdot \cos\left[\frac{\lambda_2}{L} \cdot x\right] + D_2 \cdot \cosh\left[\frac{\lambda_2}{L} \cdot x\right]$$

$$\phi_3(x) := A_3 \cdot \sin\left[\frac{\lambda_3}{L} \cdot x\right] + B_3 \cdot \sinh\left[\frac{\lambda_3}{L} \cdot x\right] + C_3 \cdot \cos\left[\frac{\lambda_3}{L} \cdot x\right] + D_3 \cdot \cosh\left[\frac{\lambda_3}{L} \cdot x\right]$$

$$\phi_0(x) := R + x \quad \text{This is the RIGID BODY mode.}$$

APPENDIX E

STATE-SPACE FORMULATION OF EQUATIONS OF MOTION FOR REDUCED-ORDER MODEL OF RIGID DISK - FLEXIBLE BEAM SYSTEM WITH SYSTEM MODES

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\begin{bmatrix}
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & -\ddot{\omega}^2 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0_2 & 1 & 0 & 0 \\
 0 & 0 & 0 & 0 & -\ddot{\omega}_2 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0_2 & 1 \\
 0 & 0 & 0 & 0 & 0 & 0 & -\dot{\omega}_3 & 0
 \end{bmatrix} = [A]$$

$$\begin{bmatrix}
 0 \\
 \frac{d}{dx} \phi(0)_0 \\
 0 \\
 \frac{d}{dx} \phi(0)_1 \\
 0 \\
 \frac{d}{dx} \phi(0)_2 \\
 0 \\
 \frac{d}{dx} \phi(0)_3
 \end{bmatrix} = [B]$$

$$\begin{bmatrix}
 \phi(L)_0 & \phi(L)_1 & \phi(L)_2 & \phi(L)_3 \\
 0 & 0 & 0 & 0 \\
 \frac{d}{dx} \phi(0)_0 & \frac{d}{dx} \phi(0)_1 & \frac{d}{dx} \phi(0)_2 & \frac{d}{dx} \phi(0)_3
 \end{bmatrix} = [C] \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} = [D]$$

Output 1: absolute beam tip position (inches)

Output 2: disk angle (radians)

Input: Torque $u(t)$ to disk

APPENDIX F

COMPLETE EQUATIONS OF MOTION IN STATE-SPACE FORM FOR THE ELECTROMECHANICAL SYSTEM

State of variable equations: $\dot{x} = Ax + Bu$

$y = Cx + Du$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\omega_1 & -2 \cdot \eta \cdot \omega_1 & 0 \\ 0 & 0 & 0 & 0 & 0_2 \\ 0 & 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{Kb1}{L1} \cdot \frac{d}{dx} \cdot \phi(0)_0 & 0 & -\frac{Kb1}{L1} \cdot \frac{d}{dx} \cdot \phi(0)_1 & 0 \\ 0 & -\frac{Kb2}{L2} \cdot \frac{d}{dx} \cdot \phi(0)_0 & 0 & -\frac{Kb2}{L2} \cdot \frac{d}{dx} \cdot \phi(0)_1 & 0 \end{bmatrix}$$

= columns 1 through 5 of [A]

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Kt1 \cdot \frac{d}{dx} \cdot \phi(0)_0 & Kt2 \cdot \frac{d}{dx} \cdot \phi(0)_0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Kt1 \cdot \frac{d}{dx} \cdot \phi(0)_1 & Kt2 \cdot \frac{d}{dx} \cdot \phi(0)_1 \\ 0 & 0 & 0 & 0 & 0 \\ -2 \cdot \eta_2 \cdot \omega_2 & 0 & 0 & Kt1 \cdot \frac{d}{dx} \cdot \phi(0)_2 & Kt2 \cdot \frac{d}{dx} \cdot \phi(0)_2 \\ 0 & 0_2 & 1 & 0 & 0 \\ 0 & -\omega_3 & -2 \cdot \eta_3 \cdot \omega_3 & Kt1 \cdot \frac{d}{dx} \cdot \phi(0)_3 & Kt2 \cdot \frac{d}{dx} \cdot \phi(0)_3 \\ -\frac{Kb1}{L1} \cdot \frac{d}{dx} \cdot \phi(0)_2 & 0 & -\frac{Kb1}{L1} \cdot \frac{d}{dx} \cdot \phi(0)_3 & -\frac{R1}{L1} & 0 \\ -\frac{Kb2}{L2} \cdot \frac{d}{dx} \cdot \phi(0)_2 & 0 & -\frac{Kb2}{L2} \cdot \frac{d}{dx} \cdot \phi(0)_3 & 0 & -\frac{R2}{L2} \end{bmatrix}$$

= columns 6
through 10
of [A]

$$\begin{bmatrix}
 0 & 0 & 0 \\
 0 & 0 & \frac{d}{dx} \cdot \phi(0)_0 \\
 0 & 0 & 0 \\
 0 & 0 & \frac{d}{dx} \cdot \phi(0)_1 \\
 0 & 0 & \frac{d}{dx} \cdot \phi(0)_2 \\
 0 & 0 & \frac{d}{dx} \cdot \phi(0)_3 \\
 \frac{Ka1}{L1} & 0 & 0 \\
 0 & \frac{Ka2}{L2} & 0
 \end{bmatrix} = [B]$$

$$\begin{bmatrix}
 \phi(L)_0 & \frac{d}{dx} \cdot \phi(0)_0 & 0 \\
 0 & 0 & \frac{d}{dx} \cdot \phi(0)_0 \\
 \phi(L)_1 & \frac{d}{dx} \cdot \phi(0)_1 & 0 \\
 0 & 0 & \frac{d}{dx} \cdot \phi(0)_1 \\
 \phi(L)_2 & \frac{d}{dx} \cdot \phi(0)_2 & 0 \\
 0 & 0 & \frac{d}{dx} \cdot \phi(0)_2 \\
 \phi(L)_3 & \frac{d}{dx} \cdot \phi(0)_3 & 0 \\
 0 & 0 & \frac{d}{dx} \cdot \phi(0)_3 \\
 0 & 0 & 0 \\
 0 & 0 & 0
 \end{bmatrix} = [C]^T$$

$$\begin{bmatrix}
 0 & 0 & 0 \\
 0 & 0 & 0 \\
 0 & 0 & 0
 \end{bmatrix} = [D]$$

Inputs: 1 - $e1(t)$ = voltage input to motor 1
 2 - $e2(t)$ = voltage input to motor 2
 3 - T_μ = Coulomb friction torque on disk

Outputs: 1 - beam tip position (inches)
 2 - disk angular position (rad)
 3 - disk angular velocity (rad/sec)

Summary of parameter values: (units: in-lbf-sec)

$E := 3.0 \cdot 10^7$	$L := 35.125$	$\rho := 0.000725$
$R := 6.0625$	$J := 0.573$	$d := 0.249$

$$R1 := 29 \, \Omega \quad R := 55 \, \Omega \quad L1 := 0.0125 \, H$$

$$L2 := 0.025 \, H \quad Ka1 := 17.84 \quad Ka2 := 17.92$$

$$Kt1 := 4.204 \, \text{in-lbf/A} \quad Kt2 := 5.511 \, \text{in-lbf/A}$$

$$Kb1 := 0.473 \, \text{V-sec/rad} \quad Kb2 := 0.620 \, \text{V-sec/rad}$$

$$\eta_1 := .01 \quad \eta_2 := .01 \quad \eta_3 := .01$$

$$T_\mu := 0.39 \, \text{in-lbf}$$

E = modulus of elasticity of beam material

L = length of beam

ρ = mass density of beam material

R = radius of rigid disk

J = mass moment of inertia of rigid disk

d = diameter of uniform beam

R1 = electrical resistance of motor 1

R2 = electrical resistance of motor 2

L1 = electrical inductance of motor 1

L2 = electrical inductance of motor 2

Ka1 = amplifier gain for motor 1

Ka2 = amplifier gain for motor 2

Kt1 = motor 1 torque constant

Kt2 = motor 2 torque constant

Kb1 = motor 1 back emf constant

Kb2 = motor 2 back emf constant

η = damping coefficient associated with each mode

T_μ = Coulomb friction torque acting on disk

ω = frequency in rad/sec

$$\phi(L)_0 := 34.9017 \quad \phi(L)_1 := -46.5615 \quad \phi(L)_2 := 55.7653 \quad \phi(L)_3 := -56.5945$$

$$\frac{d}{dx} \phi(0)_0 := 0.8474 \quad \frac{d}{dx} \phi(0)_1 := 0.9226 \quad \frac{d}{dx} \phi(0)_2 := 0.3433 \quad \frac{d}{dx} \phi(0)_3 := 0.1625$$

$$\ddot{\omega}_1 := 53.4071 \quad \ddot{\omega}_2 := 234.2559 \quad \ddot{\omega}_3 := 638.1093$$

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